Symbolic powers of some linearly presented perfect ideals of codimension 2

Zaqueu Ramos[1] Aron Simis[2]

To Wolmer Vasconcelos on his 75th birthday, for his seminal mathematical ideas

Abstract

This work is about symbolic powers of some species of linearly presented codimension 2 perfect ideals. One studies more closely the cases where the entries of the structural matrix of the ideal are general linear forms and where this matrix is a variation of the generic Hankel matrix. The main contribution of the present approach is the use of the birational theory underlying the nature of the ideal and the details of a deep interlacing between generators of its symbolic powers and the inversion factors stemming from the inverse map to the birational map defined by the linear system of generators of this ideal.

Introduction

Let $I \subset R$ denote an ideal in a Noetherian ring and let $r \geq 0$ be an integer. The rth symbolic power $I^{(r)}$ of I can be defined as the inverse image of $S^{-1}I^r$ under the natural homomorphism $R \to S^{-1}R$ of fractions, where S is the complementary set of the union of the associated primes of R/I. There is a known hesitation as to whether one should take the whole set o associated primes of R/I or just its minimal primes or even those of minimal codimension or maximal dimension. In this work we need not worry about this dilemma because the notion will only be employed in the case of a codimension 2 perfect ideal in a Cohen–Macaulay ring – actually, a polynomial ring over a field. In this setup there is no ambiguity and $I^{(r)}$ is precisely the intersection of the primary components of the ordinary power I^r relative to the associated primes of R/I, i.e., the unmixed part of I^r.

A more serious problem is the characteristic of the base field. In characteristic zero, if I is a radical ideal, one has the celebrated Zariski–Nagata differential characterization of $I^{(r)}$ (see [11, 3.9] and the references there). The subject in positive characteristic or mixed characteristic gives a quite different panorama, often much harder but with different methods anyway. In this work we assume throughout characteristic zero. This is not due to a need of using the Zariski–Nagata criterion upfront, but rather for the urge of dealing with Jacobian matrices and using Bertini's theorem. Many technical results will be valid just over an infinite field, hence there has been an effort to convey when the characteristic is an issue at specific places. On the other hand, since we will draw quite substantially on aspects of birational maps, it may be a good idea in those instances to think about k as being algebraically closed.

The main object of concern is an $m \times (m-1)$ matrix whose entries are linear forms in a polynomial ring $R = k[X_1, \ldots, X_n]$ over an infinite field k, and the ideal of our interest will be

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the ideal $I \subset R$ generated by the $(m-1)$-minors of the matrix. We will specially be considering matrices whose entries are *general* linear forms in R. In an imprecise way they will be called *general linear* matrices. The group $\mathrm{Gl}(m,k) \times \mathrm{Gl}(n,k) \times \mathrm{Gl}(m-1,k)$ acts on the set of all linear $m \times (m-1)$ matrices over k. If $n >> m$ the orbit of a general linear matrix under this action will contain only general linear matrices. But, otherwise, the orbit may contain matrices that are not 1-generic. This scrambling in the orbit of a general linear matrix is a root of difficulty when handling ideal theoretic properties stemming from the data. For this reason, one is advised to be careful while applying liberally this action in the present considerations.

Of course, any set of random k-linear combinations of the variables $X_1, \ldots, X_n$ is a general set of 1-forms. Any such set of cardinality $m(m-1)$ can be ordered as the entries of a general linear matrix over R, so there are plenty of such matrices. Variations of the notion of a general linear matrix are available. A weaker notion – *semi-general linear* matrix – would require that every subset of entries of cardinality at most n be k-linearly independent. Perhaps the most popular of these variations is the concept of a 1-*generic* linear matrix introduced in [9, 10], in which one requires that its orbit under the above action contains no matrix with some zero entry. Clearly, for an $m \times (m-1)$ matrix this condition implies that at least $n \geq m$. Though natural in various contexts, it goes only "half" way of the cases.

Note that a general linear $m \times (m-1)$ matrix with $m(m-1) < n$ is up to a linear k-automorphism just a generic matrix leaving out a set of $n - m(m-1)$ variables. For this reason, we will assume once for all that $m(m-1) \geq n$.

One may say that the idea behind the present subject is that, in contrast to the usual approach in which one is given perfectly designed discrete or algebraic data and then subjects these data to asymptotic or probabilistic ado, here one takes up the reverse procedure, by starting out with some random like definition and then pursue some well-defined algebraic behavior for these data. If one thinks about it, the apparent difficulty surfaces at once. This may explain why some of the arguments spelled in the paper are so long and detailed, whereas they intuitively appear near to obvious.

Now, for a homogeneous ideal $I \subset R$ generated in fixed degree, whose syzygies are generated by "sufficiently many" linear syzygies, its generators are very close to span a linear system defining a birational map from a projective space onto its image. This strategy has been largely explored in recent years by several authors. Thus, the details of the geometry of birational maps can be accommodated in terms of numerical invariants from commutative algebra. However, finding room in this accommodation for symbolic powers has not, to our knowledge, been brought up so far. This is one of our main observations in this work. Together with a good grip of the algebraic and homological properties of the base ideal I, it constitutes the main bulk of the paper.

The main results of this paper are shown in Theorem 2.1, Theorem 2.8, Theorem 2.13, Theorem 2.14, Theorem 2.18 and Theorem 2.22.

We now briefly describe the contents of each section.

The first section speaks for itself. Even then, it should be stressed that the first part gives the tool used to approach the nature of the symbolic algebra in the present context. It is based on an idea of Vasconcelos, who has brought in the management of the ideal transform in this setup. As for the second part, it brings up two apparently thus far unnoticed properties of the so-called inversion factor of a birational map. These properties are proved in Proposition 1.2 and Proposition 1.3.

The second section contains the main results of the paper. It starts with some preliminaries on a perfect ideal $I \subset R = k[X_1, \ldots, X_n] (n \geq 3)$ of codimension 2 whose structural matrix $m \times (m-1)$ is a general linear matrix. We first show that the other Fitting ideals attain an expected codimension and that I enjoys typical properties which depend on the values of m, n. Thus, for $n \geq 4$, I is a normal prime ideal provided $\mathrm{char}(k) = 0$ (and possibly in general); moreover, it is of linear type if (and only if) $m \leq n$ and it is normally torsionfree if and only if $m < n$. Therefore, such an ideal is only really of new substance in the case where $m \geq n$. In the sequel we show that I satisfies a generalized property of Artin–Nagata, called (G_n) and that, for any exponent r, the symbolic power $I^{(r)}$ coincides with the $(\mathbf{X})$-saturation of I^r (in other words, the unmixed part of I^r is its saturation). Since I is prime for $n \geq 4$, the symbolic power $I^{(r)}$ is just the I-primary component of I^r and the latter has at most one further associated prime, namely, $(\mathbf{X})$. This is as far as the general expectation goes.

To go one step forward, we introduce certain graded pieces of the approximation complex, along with other techniques and a recent result of A. B. Tchernev, to deduce that if $I^{(r)} \neq I^r$ then necessarily $r \geq n-1$. This result becomes an important tool for the rest of the work.

So much for the main ideal theoretical and homological properties. On a second part of the same section, we deal with the relational nature of I. Thus, we make the Rees algebra of I intervene as associated with the underlying birational map based on the linear system of the generators of I. Specifically, we show that for $m \geq n \geq 3$ the ideal I is the base ideal of a birational map of $\mathbb{P}^{n-1}$ onto the image in $\mathbb{P}^{m-1}$. This result is based on the *fiber type* nature of I – i.e., when its Rees algebra is simplest beyond the linear type situation – and on a special case of the criterion of birationality established in [8].

In this part we bring up the *inversion factors* of the birational map based on the linear system of generators of I and show that they are natural elements in the symbolic power $I^{(n-1)}$ not belonging to the ordinary power I^{n-1}. We succeed in going this far for general values $m \geq n \geq 3$. Inversion factors have appeared before in the classical theory of plane Cremona maps, where they are a version of the *principal curves* (see, e.g., [1, Chapter 3]). However, to our knowledge the notion has never been explicitly addressed for Cremona maps in higher dimension, much less for birational maps onto their images (classically called "rational representations" of projective space). We introduce them here in this larger generality and dimension. A bit surprisingly, they keep in certain cases a strong relation to a Jacobian determinant – so to say, an analogue of the relationship between principal curves and factors of the classical *Jacobian curve* (see, e.g., Proposition 2.11). Our main interest here in these inversion factors is the significant role they play as regards the generation of some symbolic powers.

To thrive deeper, we assume that either $m = n$ (the "Cremona case") or $m = n+1$ (the "implicitization case"). Our main drive is to tell the precise structure of the symbolic algebra $\mathcal{R}^{(I)}$ of I. When $m = n$ our main results follows by drawing on some of the results of the earlier subsections and collecting various pieces throughout the previous literature. The main result says that $\mathcal{R}^{(I)}$ is generated in degrees 1 and $n-1$, with only one fresh generator in degree $n-1$ which may taken to be the source inversion factor of the Cremona map defined by the n-minors of $\mathcal{L}$. Moreover, in characteristic zero, this generator coincides up to a scalar with the Jacobian determinant of those same minors.

The case $m = n+1$ requires a full tour de force across the material and does not follow straightforwardly from the previously stated results in the paper. First, the generation of $\mathcal{R}^{(I)}$ is more involved, occurring in degrees 1, $n-1$ and $n(n-1)-1$. This time around, showing that the source inversion factors constitute a minimal set of fresh generators in degree $n-1$ is far from

straightforward. Here we resorted to local duality as applied to $\mathrm{H}^0_{(\mathbf{X})}(R/I^{n-1}) \simeq I^{(n-1)}/I^{n-1}$ and to a subtle result on the R-dual of the last nonfree syzygy module in the minimal free resolution of R/I^{n-1}. The argument here depends strongly on the basic assumption that I is the ideal of n-minors of a matrix whose entries are general linear forms - the result totally crumbles down for matrices with linear entries which are not sufficiently general. In fact, even deformations of such linear matrices may not lead to the strong results obtained here. Actually, a given deformation of a linear matrix may not even keep the basic properties of the ideal of maximal minors, such as radicalness or primality.

This is the first step. In order to advance into proving the generation of the symbolic algebra we describe a set of generators of its defining ideal, much in the spirit of [24, Sections 5–8], but quite a bit more involved. Making these generators explicit forced us to uncover a whole world of very tight relation between the various constructs coming from the melange of symbolic power and birational theories. A particular aspect that makes a case for this assertion is the long proof required to show that a certain variable is not a zerodivisor modulo the ideal generated by the "expected" symbolic relations (proof of Theorem 2.22). We have applied Gröbner basis theory via a case-by-case S-polynomial analysis in which the conclusions depend strongly on the theoretical material developed before. Thus, it is not really the algorithm that matters, but rather the use of the previous theory as a quality control. Due to the amount of technical passages, we refer the reader to the appropriate places in the paper.

The last section takes a detour to look at a codimension perfect ideal $I \subset R$ whose supporting matrix, though not general linear in the previous sense, has a more structured nature which compensates for the lack of general linearity. The idea is to build suitable heiresses or modifications of the generic Hankel matrix. We examined three of these constructs: catalecticants with leap, sub-Hankel and quasi-Hankel generic matrices. The results are close to the ones in the general linear case, but there are some unexpected differences. Thus, for example, in the sub-Hankel case with $\dim R \geq 4$, the ring R/I is a non-normal domain. Moreover, by construction only the quasi-Hankel case leads to a Cremona transformation. The Cremona transformations obtained this way do not seem to have been observed before in this systematic way.

1 Terminology

1.1 Generalities on symbolic powers

We will assume throughout that $R = k[X_1, \ldots, X_n]$ is a standard graded polynomial ring over an infinite field k. Given an ideal $I \subset R$ and an integer $r \geq 1$, the rth *symbolic power* $I^{(r)}$ of I is the contraction of $S^{-1}I^r$ under the natural homomorphism $R \to S^{-1}R$ of fractions, where S is the complementary set of the union of the associated primes of R/I. In this work I will be a codimension 2 perfect ideal, hence R/I is Cohen–Macaulay and so I is a pure (unmixed) ideal. In this setup then $I^{(r)}$ is precisely the intersection of the primary components of the ordinary power I^r relative to the associated primes of R/I, i.e., the unmixed part of I^r.

A slightly different way to envisage symbolic powers is by noting that the $(I^{(r)} \cap I^{r-1})/I^r$ is the R/I-torsion of the conormal module I^{r-1}/I^r of order r. Taking the direct sum over all $r \geq 0$ yields the R/I-torsion of the associated graded ring of I, hence the non triviality of symbolic powers gives a measure of the torsion of the latter. In particular, there is no nonzero torsion if and only if $I^{(r)} = I^r$ for every $r \geq 0$ – in which case one says that the ideal I is *normally torsionfree*. However, this information is most of the times pretty useless once it holds. What

matters for a substantial class of ideals – codimension 2 perfect ones included – is to guess some sort of asymptotic behavior for the equality of the two powers, more like an "inf-asymptotic" such behavior in the sense that one has equality throughout up to a certain exponent order, thereafter comparison gets disorganized or even chaotic.

We observe that, like the ordinary powers, the symbolic powers constitute a decreasing multiplicative filtration, so one can consider the corresponding *symbolic Rees algebra* $\mathcal{R}_R^{(I)} = \bigoplus_{r\geq 0} I^{(r)}t^r \subset R[t]$. However, unlike the ordinary Rees algebra, this algebra may not be finitely generated over R. Alas, there are no definite effective ways to check when $\mathcal{R}_R^{(I)}$ is Noetherian. The necessary and sufficient conditions of Huneke ([15, Theorems 3.1 and 3.25]) obtained in dimension 3 are not effective and neither is the necessary condition of Cowsik–Vasconcelos ([7], [21, Proposition 3.5]). Nevertheless, the latter becomes quite effective provided one has a good guess about what finitely generated subalgebra is a strong candidate. In a precise way, one has the following strategy.

First recall that, given an ideal $I \subset R$, where R is a Noetherian domain with field of fractions K, the *ideal transform* of R relative to I is the R-subalgebra $T_R(I) := R :_K I^\infty \subset K$. We will draw on the following two fundamental facts:

- ([24, Proposition 7.1.4]) If $C \subset T_R(I)$ is a finitely generated R-subalgebra such that $\mathrm{depth}_{IC}(C) \geq 2$ then $C = T_R(I)$.
- ([24, Proposition 7.2.6]) If R moreover satisfies the condition (S_2) of Serre then
$$\mathcal{R}_R^{(I)} \simeq T_{\mathcal{R}(I)}(J) \subset R[t]$$
as R-subalgebras of $R[t]$ for suitable choice of the ideal $J \subset R$.

Our idea of applying these principles is summarized in the following result, of immediate verification:

Proposition 1.1. *Let $R = k[X_1, \ldots, X_n]$ denote a standard graded polynomial ring over an infinite field k, with irrelevant maximal ideal $(\mathbf{X}) := (X_1, \ldots, X_n)$. Let $I \subset R$ stand for a homogeneous ideal satisfying the following properties:*

(i) *For every $r \geq 0$, the R-module $I^{(r)}/I^r$ is either zero or $(\mathbf{X})$-primary.*

(ii) $\mathrm{depth}_{(\mathbf{X})C}(C) \geq 2$ *for some finitely generated graded R-subalgebra $C \subset \mathcal{R}_R^{(I)}$ containing the Rees algebra $\mathcal{R}_R(I)$.*

Then $C = \mathcal{R}_R^{(I)}$.

We observe that the typical graded R-subalgebra $C \subset \mathcal{R}_R^{(I)}$ containing the Rees algebra $\mathcal{R}_R(I)$ as above has the form $C = R[It, I^{(2)}t^2, \ldots, I^{(s)}t^s] \subset R[t]$, for suitable $s \geq 1$. Although the non-vanishing of certain of the R-modules $I^{(r)}/I^r$ gives a measure of how far one has to go (provided the symbolic Rees algebra is finitely generated), it is really the R-modules
$$\frac{I^{(r)}}{\sum_{1\leq j\leq r-1} I^{(r-j)} \cdot I^{(j)}}$$
that count for the search of *fresh* (or *genuine*) generators of the algebra. Although this is a well-known simple observation, it often encrypts some subtleties in a particular case.

1.2 Birational maps and inversion factors

Our reference for the basics in this part is [19], which contains enough of the introductory material in the form we use here (see also [8] for a more general overview).

Let k denote an arbitrary infinite field – further assumed to be algebraically closed in a geometric discussion. A rational map $\mathfrak{G} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{m-1}$ is defined by m forms $\mathbf{g} = \{g_1, \ldots, g_m\} \subset R := k[\mathbf{X}] = k[X_1, \ldots, X_n]$ of the same degree $d \geq 1$, not all null. We often write $\mathfrak{G} = (g_1 : \cdots : g_m)$ to underscore the projective setup and assume that $\gcd\{g_1, \cdots, g_m\} = 1$ (in the geometric terminology, the linear system defining $\mathfrak{G}$ "has no fixed part"), in which case we call d the *degree* of $\mathfrak{G}$.

Although the the definition of the rational map $\mathcal{G}$ depends on the linear system spanned by the defining coordinates, its scheme theoretic indeterminacy locus is defined by the ideal of R generated by the members of this system. For convenience, this ideal will slightly improperly be referred to as the *base ideal* of $\mathcal{G}$.

The image of $\mathfrak{G}$ is the projective subvariety $W \subset \mathbb{P}^{m-1}$ whose homogeneous coordinate ring is the k-subalgebra $k[\mathbf{g}] \subset R$ after degree renormalization. Write $k[\mathbf{g}] \simeq k[\mathbf{Y}]/I(W)$, where $I(W) \subset k[\mathbf{Y}] = k[Y_1, \ldots, Y_m]$ is the homogeneous defining ideal of the image in the embedding $W \subset \mathbb{P}^{m-1}$.

We say that $\mathfrak{G}$ is *birational onto the image* if there is a rational map $\mathbb{P}^{m-1} \dashrightarrow \mathbb{P}^{n-1}$ with defining coordinate forms $\mathbf{f} = \{f_1, \ldots, f_n\} \subset k[\mathbf{Y}]$ (not simultaneously vanishing modulo $I(W)$) satisfying the relations

$$(\mathbf{f}_1(\mathbf{g}) : \cdots : \mathbf{f}_n(\mathbf{g})) = (X_1 : \cdots : X_n),\ (\mathbf{g}_1(\mathbf{f}) : \cdots : \mathbf{g}_m(\mathbf{f})) \equiv (Y_1 : \cdots : Y_m) \pmod{I(W)}$$

Let K denote the field of fractions of $k[\mathbf{g}]$. The coordinates $\{f_1, \cdots, f_n\}$ defining the "inverse" map are not uniquely defined; any other set $\{f'_1, \cdots, f'_n\}$ inducing the same element of the projective space $\mathbb{P}^{n-1}_K = \mathbb{P}^n_k \otimes_k \mathrm{Spec}(K)$ will do as well – both tuples are called *representatives* of the rational map. Furthermore, one can choose a finite minimal set $\mathbf{f_1}, \ldots, \mathbf{f_s}$ of these representatives such that any other representative belongs to the $k[\mathbf{Y}]/I(W)$-submodule generated by $\mathbf{f_1}, \ldots, \mathbf{f_s}$. More exactly, any such a minimal representative is the transpose of a minimal generator of the syzygy module of the so-named weak Jacobian dual matrix (for complete details see [19], particularly Proposition 1.1 and [8, Section 2]). Such a set will be referred to in the sequel as *a complete set of minimal representatives* of the inverse map.

Having information about the inverse map – e.g., about its degree – will be quite relevant in the sequel. For instance, the structural congruence

$$(f_1(g_1, \ldots, g_m), \ldots, f_n(g_1, \ldots, g_m)) \equiv (X_1, \ldots, X_n) \tag{1}$$

involving the inverse map, in terms of one of its representatives lifted to $k[\mathbf{y}]$, gives a uniquely defined form $D \in R$ such that $f_i(g_1, \ldots, g_m) = X_i D$, for every $i = 1, \ldots, n$. We call D the *source inversion factor* of $\mathfrak{G}$ associated to the given representative. Clearly, there is a counterpart *target inversion factor*, defined in a similar way by exchanging the roles of $\mathbf{f}$ and $\mathbf{g}$ in (1).

A fundamental property of the inversion factor does not seem to have been observed before in the following generality and explicitness.

Proposition 1.2. *Let $\mathfrak{G}$ denote a Cremona map of $\mathbb{P}^{n-1}$ defined by forms $\mathbf{g} : \{g_1, \ldots, g_n\}$ in R without fixed part and let $\Theta(\mathbf{g})$ denote the Jacobian matrix of $\mathbf{g}$. Then* $\det(\Theta(\mathbf{g}))$ *divides a power of the source inversion factor G of $\mathfrak{G}$. In particular, if* $\det(\Theta(\mathbf{g}))$ *is reduced then it divides G.*

Proof. Let $\mathbf{f} : \{f_1, \ldots, f_n\}$ define the inverse map. Applying the chain rule of derivatives to the structural equation $\mathbf{f}(\mathbf{g}) = G \cdot (\mathbf{X})$, it obtains

$$\Theta(\mathbf{f})(\mathbf{g}) \cdot \Theta(\mathbf{g}) = G \cdot \mathcal{I} + (\mathbf{X})^t \cdot \mathrm{Grad}(G) \tag{2}$$

where $\mathcal{I}$ is the identity matrix and $\mathrm{Grad}(G) = (\partial G/\partial X_1 \ldots \partial G/\partial X_n)$. Note that the right side of (2) is the result of evaluating $\lambda \mapsto G$ in the characteristic matrix $\lambda\mathcal{I} - \mathcal{A}$, where $\mathcal{A} = -(\mathbf{X})^t \cdot \mathrm{Grad}(G)$.

Recall that, quite generally the characteristic polynomial $p(t) = t^n + a_1 t^{n-1} + \ldots + a_{n-1}t + a_n$ of a matrix $\mathcal{A}$ can be recursively computed as:

$$\begin{aligned} -a_1 &= s_1 \\ -ra_r &= s_r + \sum_{i=1}^{r-1} s_i a_{r-i} \end{aligned} \tag{3}$$

where s_r is the trace of the matrix $\mathcal{A}^r$, for $1 \leq r \leq n$.

Since

$$\mathrm{Trace}((\mathbf{X})^t \cdot \mathrm{Grad}(G)) = \sum_{i=1}^{n} X_i \frac{\partial G}{\partial X_i} = (n(n-1)-1)G \tag{4}$$

and

$$((\mathbf{X})^t \cdot \mathrm{Grad}(G))^r = G^{n-1} \cdot (\mathbf{X})^t \cdot \mathrm{Grad}(G) \tag{5}$$

it follows from (3) that $p(t) = t^n + (n(n-1)-1)Gt^{n-1}$.

Therefore, computing determinants in (2) yields

$$\det(\Theta(\mathbf{f})(\mathbf{g})) \cdot \det(\Theta(\mathbf{g})) = n(n-1)G^n. \tag{6}$$

Therefore, $\det(\Theta(\mathbf{g}))$ indeed divides G^n. $\square$

From the other end, the basic result relating inversion factors to symbolic powers is the following proposition – it too does not seem to have been been explicitly pointed out in the previous literature.

Proposition 1.3. *Let $I \subset R = k[\mathbf{x}]$ denote the base ideal of a birational map $\mathfrak{F} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{m-1}$ onto the image, satisfying the canonical restrictions. Let $D \subset R$ denote the source inversion factor relative to a given minimal representative of the inverse map. Suppose that I is a radical ideal such that $I^{(\ell)} = I^\ell$ for every $\ell \leq d'-1$, but $I^{(d')} \neq I^{d'}$, where d' is the degree of the coordinates of the representative. Then*

(a) *D is a genuine symbolic element of order d'*

(b) *Moreover, if $I^{(d')}$ is generated in standard degree $\geq dd'-1$, where d is the degree of the coordinates of $\mathfrak{F}$, then D is a homogeneous minimal generator of the symbolic Rees algebra.*

Proof. The characteristic property of D is the congruence (1). In particular, $D \in I^{d'} : (\mathbf{x})$. We may assume that $\mathfrak{F}$ is not the identity map of $\mathbb{P}^{n-1}$. Since I is radical, it has codimension at most $\dim R - 1$; hence, there is a form $h \in (\mathbf{x}) \setminus P$, for every minimal prime P of R/I, such that $hD \in I^{d'}$. This means that $D \in I^{(d')}$.

Part (a) follows straightforwardly because, under the hypothesis that $I^{(\ell)} = I^\ell$, $\ell \leq d'-1$, being genuine just means not belonging to the ordinary power I^r. Part (b) is clear since $\deg(D) = dd'-1$. $\square$

2 Ideals of general linear forms

2.1 Arithmetic and homological properties

Let k stand for an infinite field and let $R = k[X_1, \ldots, X_n]$ denote a standard graded polynomial ring over k. Quite often we will require that $\mathrm{char}(k) = 0$, but some of the results will be valid in any characteristic.

In this part we deal with matrices $m \times (m-1)$ whose entries are general linear forms in R. In an imprecise way they will be called *general linear* matrices.

Assuming $m(m-1) \geq n$, any general linear $m \times (m-1)$ matrix $\mathcal{L} = (\ell_{ij})$ over $R = k[X_1, \ldots, X_n]$ is a specialization of the generic $m \times (m-1)$ matrix $\mathbf{Z} = (Z_{i,j})$ over the ring R via the regular sequence $\{Z_{i,j} - \ell_{i,j} | i, j\}$. Here we will be interested in specializing from $\mathbf{Z} = (Z_{ij})$ rather over the field k. Moreover, one wishes that the ideal of $(m-1)$-minors specialize accordingly.

Following common usage, one denotes by $I_t(\Psi) \subset R$ the ideal generated by the $t \times t$ minors of a matrix Ψ. The basic result is then:

Theorem 2.1. *Let $\mathcal{L}$ denote an $m \times (m-1)$ general linear matrix over $R = k[X_1, \ldots, X_n]$, with $n \geq 3$ and $m \geq 2$. Then, for every $1 \leq t \leq m-1$, one has*

$$\mathrm{ht}\,(I_t(\mathcal{L})) = \min\{n, (m-t+1)(m-t)\}.$$

Proof. We use the same notation for the matrix as for the set of its entries. Thus, let $\mathbf{Z} = (Z_{i,j})$ be the generic $m \times (m-1)$ matrix over k and $S := k[\mathbf{Z}]$. Similarly, let $k[\mathcal{L}] \subset R$ denote the subalgebra generated over k by the entries $\{\ell_{i,j}\}$ of $\mathcal{L}$. Write $\mathbf{L}$ for the kernel of the k-algebra homomorphism $S \to R$ mapping $Z_{i,j} \mapsto \ell_{i,j}$. Then $\mathbf{L}$ is generated by $s := m(m-1) - n$ linear forms in S. Clearly, for any $1 \leq t \leq m-1$, the ideal $(I_t(\mathbf{Z}), \mathbf{L})/\mathbf{L}$ maps to the ideal $I_t(\mathcal{L}) \subset R$ under the injection $S/\mathbf{L} \hookrightarrow R$, which allows for a specialization argument.

So far, these remarks work for any linear matrix. But if $\mathcal{L}$ is general we clearly have $R = k[\mathcal{L}]$, hence $S/\mathbf{L} \simeq R$. By a similar token, $\mathbf{L}$ is generated by general linear forms, say, $\mathbf{L} := (L_1, \ldots, L_s)$. (One can actually be a lot more explicit, though not needed for the argument. Namely, in the revlex monomial order with $Z_{1,1} > \cdots > Z_{1,m-1} > Z_{2,1} \cdots > Z_{m,m-1}$, the initial ideal of $\mathbf{L}$ is generated by the first $m(m-1)-n$ variables in this order. Therefore, $\mathbf{L}$ is generated by 1-forms $L_{i,j} := Z_{i,j} - \lambda(i,j)$, where $\{Z_{i,j}\}$ are the first $m(m-1) - n$ variables and $\lambda(i,j)$ is a general 1-form depending only on the last n variables).

Because $S/(I_t(\mathbf{Z}), \mathbf{L}) \simeq R/I_t(\mathcal{L})$ by specialization, the assertion is equivalent to having

$$\dim S/I_t(\mathbf{Z}), \mathbf{L}) = \max\{0, n - (m-t+1)(m-t)\}. \tag{7}$$

Recall that in the generic case one has $\dim S/I_t(\mathbf{Z}) = m(m-1) - (m-t+1)(m-t)$. For convenience, we will write $D := m(m-1) - (m-t+1)(m-t)$.

We proceed by induction on $\mu(\mathbf{L}) = m(m-1) - n$. Obviously, $D > 0$ if and only if $t \geq 2$. Now, for every t in this range, clearly L_1 is a non-zero-divisor on $S/I_t(\mathbf{Z})$ since L_1 is a linear form and all the associated primes are contained in the one single prime $I_2(\mathbf{Z})$ generated in degree 2. Therefore, one has

$$\dim S/(I_t(\mathbf{Z}), L_1) = \begin{cases} 0 & \text{if } t = 1 \\ D - 1 & \text{otherwise} \end{cases}$$

Let now $m(m-1)-n \geq j \geq 2$. By the inductive hypothesis, one has

$$\dim S/(I_t(\mathbf{Z}), L_1, \ldots, L_{j-1}) = \begin{cases} 0 & \text{if } D \leq j-1 \\ D-(j-1) & \text{otherwise} \end{cases}$$

Letting $J_{[1]}$ denote the part of degree 1 of a homogeneous ideal J in R, since L_j is a general form we have $L_j \notin P_{[1]}$ for every prime $P \in \bigcup_t \mathrm{Ass}(S/I_t(\mathbf{Z}), L_1, \ldots, L_{j-1})$, with t in the range $D > j-1$. Then the dimension again drops by 1, i.e., we get

$$\dim S/(I_t(\mathbf{Z}), L_1, \ldots, L_j) = \begin{cases} 0 & \text{if } D \leq j \\ D-j & \text{otherwise} \end{cases}$$

Applying with $j = m(m-1)-n$ yields

$$\dim S/(I_t(\mathbf{Z}), \mathbf{L}) = \begin{cases} 0 & \text{if } D \leq m(m-1)-n \\ D-m(m-1)-n & \text{if } D > m(m-1)-n \end{cases}$$

Retrieving the value of D yields the required result. $\square$

Remark 2.2. The above proof used the property that the linear form L_j avoids the degree 1 part of associated primes of earlier specializations. One might think that this property alone could thereof be taken to mean "general linear form". However, later on we will use other strong features of general linear forms that do not follow from this property.

Proposition 2.3. *Let $\mathcal{L}$ denote an $m \times (m-1)$ general linear matrix over $R = k[X_1, \ldots, X_n]$, with $n \geq 3$ and $m \geq 2$. Set $I = I_{m-1}(\mathcal{L}) \subset R$. Then:*

(i) *I has codimension* 2 *and $I_{m-2}(\mathcal{L}) \subset R$ has codimension* $\min\{6, n\}$.

(ii) $(\mathrm{char}(k) = 0)$ *R/I satisfies the condition (R_r) of Serre, with* $r = \min\{3, n-2-1\}$; *in particular, if $n \geq 4$ then R/I is normal and I is a prime ideal.*

(iii) *I is of linear type if and only if $m \leq n$.*

(iv) $(\mathrm{char}(k) = 0)$ *I is normally torsionfree if and only if $m < n$.*

Proof. (i) This follows from Theorem 2.1.

(ii) Since $\mathbf{Z}$ is a generic matrix, the Jacobian ideal of $S/I_{m-1}(\mathbf{Z})$ is $I_{m-2}(\mathbf{Z})/I_{m-1}(\mathbf{Z})$. Applying Bertini's theorem ([13]) gives that the singular scheme of the scheme-theoretic general hyperplane section $S/(I_{m-1}(\mathbf{Z}), L_1)$ is the scheme associated to $S/(I_{m-2}(\mathbf{Z}), L_1)$. Inducting on the number $m(m-1)-n$ of general hyperplane sections yields that the singular scheme of the scheme-theoretic linear section $S/(I_{m-1}(\mathbf{Z}), \mathbf{L}) \simeq R/I$ is the scheme associated to $S/(I_{m-2}(\mathbf{Z}), \mathbf{L}) \simeq R/I_{m-2}(\mathcal{L})$. By Theorem 2.1, the latter has codimension at least $\min\{6, n\}$ on R. Since $I = I_{m-1}(\mathcal{L})$ has codimension 2, R/I satisfies the Serre condition $(R_{\min\{3,n-2-1\}})$. Thus, if $n \geq 4$ then R/I satisfies (R_1). At the other end, R/I is Cohen–Macaulay. It follows that, for $n \geq 4$, R/I is normal and, since I is homogeneous, R/I must be a domain. (If $n = 3$ then I is still a radical ideal.)

(iii) Let us apply the result of Theorem 2.1 in this case. We claim that

$$\min\{n, (m-t+1)(m-t)\} \geq m-t+1, \text{ for } 1 \leq t \leq m-1.$$

This is obvious if the minimum is attained by $(m-t+1)(m-t)$; if the minimum is n instead then $m \leq n$ certainly implies $m-t+1 \leq n$.

This shows that I satisfies the property (F_1), hence it is an ideal of linear type in this case (see [14]). The converse is evident since the linear type property implies the inequality $\mu(I) \leq \dim R$.

(iv) Suppose first that $m < n$. By part (iii), I is of linear type. Since I is strongly Cohen–Macaulay ([3, Theorem 2.1(a)]) then the Rees algebra of I is Cohen–Macaulay ([14, Theorem 9.1]), and hence so is the associated graded ring of I. On the other hand, we may assume that $n \geq 4$ given that for $n = 3$ the ideal I is generated by a regular sequence of two elements. Therefore, by part (ii), the ideal I is prime. By [12, Proposition 3.2 (1)], the assertion is equivalent to having

$$\ell_P(I) \leq \max\{\mathrm{ht}\, P - 1, \mathrm{ht}\, I\},$$

for every prime ideal $P \supset I$. Since I is homogeneous, it suffices to take P homogeneous. We may assume that $\mathrm{ht}\, P \geq 3$ since I is a height 2 prime. Therefore, we have to show that $\ell_P(I) \leq \mathrm{ht}\, P - 1$. If $P = (\mathbf{X})$ the result is clear since $\ell_{(\mathbf{X})} \leq \mu(I) = m \leq n-1 = \mathrm{ht}\,(\mathbf{X}) - 1$. Therefore, we may assume that $P \subsetneq (\mathbf{X})$, hence $\mathrm{ht}\, P \leq n-1$.

Set $t_0 := \max\{1 \leq s \leq m-2 \,|\, I_s(\mathcal{L}) \not\subset P\}$. Therefore, $I_{t_0+1}(\mathcal{L}) \subset P$, hence $\mathrm{ht}\, I_{t_0+1}(\mathcal{L}) \leq \mathrm{ht}\, P \leq n-1$. By Theorem 2.1 one must have $\mathrm{ht}\, I_{t_0+1}(\mathcal{L}) = (m-t_0)(m-t_0-1)$. Pick a t_0-minor Δ of $\mathcal{L}$ not contained in P, so that, in particular, R_P is a localization of the ring of fractions $R_\Delta = R[\Delta^{-1}] \subset k(\mathbf{X})$. By a standard row-column elementary operation procedure, there is an $(m-t_0) \times (m-t_0-1)$ matrix $\widetilde{\mathcal{L}}$ over R_P such that

$$I_P = I_{m-1-t_0}(\widetilde{\mathcal{L}}).$$

Assume first that $t_0 \leq m-3$. Then $(m-t_0) \leq (m-t_0)(m-t_0-1)-1 = \mathrm{ht}\, I_{t_0+1}(\mathcal{L})-1 \leq \mathrm{ht}\, P-1$. Therefore

$$\ell_P(I) = \ell(I_{m-1-t_0}(\widetilde{\mathcal{L}})) \leq \min\{\mu(I_{m-1-t_0}(\widetilde{\mathcal{L}})), \mathrm{ht}\, P\} = \min\{m-t_0, \mathrm{ht}\, P\} \leq \mathrm{ht}\, P - 1.$$

If $t_0 = m-2$, one gets $\ell_P(I) = \min\{2, \mathrm{ht}\, P\} = 2 \leq \mathrm{ht}\, P - 1$ since it has been assumed that $\mathrm{ht}\, P \geq 3$.

Therefore, I is normally torsionfree. The converse will follow from Theorem 2.10 (a). $\square$

The proof of the main theorem stated further down will draw on several results of independent interest.

Recall the following notation: for a given integer $s \geq 1$, one says that the ideal $I \subset R$ satisfies condition (G_s) if $\mu(I_P) \leq \mathrm{ht}\, P$, for every prime ideal P such that $\mathrm{ht}\, P \leq s-1$.

Proposition 2.4. *Let $\mathcal{L}$ denote an $m \times (m-1)$ general linear matrix over $R = k[X_1, \ldots, X_n]$, with $m \geq 2$ and $n \geq 3$. Set $I := I_{m-1}(\varphi)$. Then*

(i) *I satisfies condition (G_n).*

(ii) (char$(k) = 0$) *Given an integer $r \geq 0$ such that $I^{(r)}/I^r \neq \{0\}$ then $I^{(r)}/I^r$ is an $(\mathbf{X})$-primary R-module (in other words, $I^{(r)}$ is the saturation of I^r).*

Proof. (i) Let $P \subset R$ be a prime of height $\leq n-1$. Set

$$t_\infty := \min\{1 \leq t \leq m-1 \,|\, I_t(\mathcal{L} \subset P)\}.$$

Then $\operatorname{ht} I_{t_\infty}(\mathcal{L}) \leq n-1$, hence $\operatorname{ht} I_{t_\infty}(\mathcal{L}) = (m-t_\infty+1)(m-t_\infty)$ by Proposition 2.3 (i). Inverting a $(t_\infty - 1)$-minor of $\mathcal{L}$ in R_P we get $I_P = I_{m-t_\infty}(\widetilde{\mathcal{L}})$ for a suitable $(m-t_\infty+1)(m-t_\infty)$ matrix $\widetilde{\mathcal{L}}$ over R_P. Collecting the information yields

$$\begin{aligned} \mu(I_P) &= \mu(I_{m-t_\infty}(\widetilde{\mathcal{L}})) = m - t_\infty + 1 \leq (m - t_\infty + 1)(m - t_\infty) \\ &= \operatorname{ht} I_{t_\infty}(\mathcal{L}) \leq \operatorname{ht}(P). \end{aligned}$$

(ii) Fixing an $r \geq 0$, suppose that $I^{(r)}/I^r \neq \{0\}$. By Proposition 2.3 (iv), we have $m \geq n$. The assertion is equivalent to saying that a power of $(\mathbf{X})$ annihilates $I^{(r)}/I^r$ i.e., that ${I^{(r)}}_P = {I^r}_P$ for every prime $P \neq (\mathbf{X})$. Letting $r \geq 0$ run, this is in turn equivalent to claiming that the associated graded ring $\operatorname{gr}_I(R)$ is torsionfree over R/I locally on the punctured spectrum $\operatorname{Spec}(R) \setminus (\mathbf{X})$.

Thus, let $P \neq (\mathbf{X})$ be a prime containing I. Then the condition (G_n) of part (i) implies that I_P satisfies the condition (F_1) (same as (G_∞)) as an ideal of R_P. As in the proof of Proposition 2.3 (iv), we know that the associated graded ring $\operatorname{gr}_{I_P}(R_P)$ is Cohen–Macaulay. Therefore, by the same token and since $\operatorname{ht} I = 2$, one has to show the local estimates

$$\ell_Q(I) = \ell_{Q_P}(I_P) \leq \operatorname{ht}(Q_P) - 1 = \operatorname{ht} Q - 1,$$

for every prime $Q \subset P$.

Fixing such a prime Q, set $t_0 := \max\{1 \leq s \leq m-2 \,|\, I_s(\mathcal{L}) \not\subset Q\}$. The argument is now the same as the one in the proof of Proposition 2.3 (iv). $\square$

Corollary 2.5. *Let $\mathcal{L}$ denote an $m \times (m-1)$ general linear matrix over $R = k[X_1, \ldots, X_n]$, with $m \geq 2$ and $n \geq 3$. Set $I := I_{m-1}(\varphi)$. Then $\mathcal{S}_r(I) \simeq I^r$ in the range $1 \leq r \leq n-1$.*

Proof. This follows from Proposition 2.4 (i) as applied through the result of [23, Theorem 5.1]. $\square$

For an integer in the range $1 \leq r \leq n-3$, recall the rth approximation complex associated to the ideal I (see [24, Section 3]):

$$\mathcal{M}_r : 0 \to H_r \to H_{r-1} \otimes S_1 \to \cdots \to H_1 \otimes S_{r-1} \to S_r. \tag{8}$$

Here H_i stands for the ith Koszul homology module on the generators of I and S_i denotes the ith homogeneous part of the polynomial ring $S := R/I[Y_1 \ldots, Y_m]$. One has $H_0(\mathcal{M}_r) \simeq \mathcal{S}_r(I/I^2)$.

Proposition 2.6. *The approximation complex $\mathcal{M}_r$ is acyclic in the range $1 \leq r \leq n-3$.*

Proof. We show that the complex is acyclic locally everywhere. Suppose first that $P \neq (\mathbf{X})$ is a non-irrelevant prime. In this case, using Corollary 2.5, the result is contained in [14, Theorem 5.1].

Thus, we can assume that $P = (\mathbf{X})$ and that $\mathcal{M}_r$ is acyclic locally at any prime properly contained in $(\mathbf{X})$. We show acyclicity stepwise from the left. Thus, suppose the partial complex

$$\begin{tikzcd} 0 \arrow[r] & H_r \arrow[r] & \cdots \arrow[r] & H_{k+2} \otimes_{r-k-2} \arrow[r] & H_{k+1} \otimes S_{r-k-1} \arrow[dr] & & \\ & & & & & B_k \arrow[dr] & \\ & & & & & & 0 \end{tikzcd}$$

is exact. Since I is a strongly Cohen–Macaulay ideal ([3, Theorem 2.1(a)]), one has depth $(H_i) = n-2$ for every $1 \leq i \leq r$. Chasing depths from left to right, one gets depth $(B_k) \geq n-(r-k+1) = (n-r)+k-1 \geq 3+k-1 = k+2 \geq 2$.

Now, letting $Z_k \subset H_k \otimes S_{r-k}$ denote the subsequent module of cycles, write $D_k := Z_k/B_k$. Suppose $D_k \neq 0$ and take $Q \in \mathrm{Ass}(D_k)$. Since the entire complex is acyclic off $(\mathbf{X})$, we must have $Q = (\mathbf{X})$. Applying $\mathrm{Hom}_R(R/(\mathbf{X}), -)$ yields the exact sequence

$$0 = \mathrm{Hom}_R(R/(\mathbf{X}), Z_k) \to \mathrm{Hom}_R(R/(\mathbf{X}), D_k) \to \mathrm{Ext}\,^1_R(R/(\mathbf{X}), B_k).$$

The rightmost term of this sequence vanishes as well since the depth of B_k is at least 2, hence also does the middle term; this is absurd since $(\mathbf{X})$ is an associated prime of D_k. Therefore, we conclude that $D_k = 0$. $\square$

Denote by $\mathrm{pd}_R(M)$ the projective dimension of a finitely generated R-module M.

Corollary 2.7. $\mathrm{pd}_R(I^r/I^{r+1}) \leq r+2$ *for* $1 \leq r \leq n-3$. *In particular,* $(\mathbf{X}) \notin \mathrm{Ass}(I^r/I^{r+1})$.

Proof. Since (8) is acyclic by Proposition 2.6, depth chasing all the way to the right yields depth $\mathcal{S}_r(I/I^2) \geq n-(r+2)$. Therefore, $\mathrm{pd}_R(\mathcal{S}_r(I/I^2)) \leq r+2$. But $\mathcal{S}_r(I/I^2) \simeq I^r/I^{r+1}$ by Corollary 2.5. $\square$

Theorem 2.8. $(\mathrm{char}(k) = 0)$ $\mathrm{Ass}(R/I^r) = \mathrm{Ass}(R/I)$ *for* $1 \leq r \leq n-2$.

Proof. By Proposition 2.4 (ii), $\mathrm{Ass}(R/I^r) \subset \mathrm{Ass}(R/I) \cup \{(\mathbf{X})\}$ - note that the assumption that I is prime holds for $n \geq 4$; for $n = 3$, I is still radical, hence the statement is obvious directly.

Proceed by induction on r. It is clear for $r = 1$ since I is a radical unmixed ideal for $n \geq 3$ and ht $(I) = 2 < n$.

Supposing $(\mathbf{X}) \in \mathrm{Ass}(R/I^r)$, the exact sequence $0 \to I^{r-1}/I^r \to R/I^r \to R/I^{r-1} \to 0$ and the inductive hypothesis force us to conclude that $(\mathbf{X}) \in \mathrm{Ass}(I^{r-1}/I^r)$. But since $r+1 \leq n-1$, the latter is forbidden by Corollary 2.7. $\square$

2.2 The role of the inverse factor

An ideal $I \subset R$ generated by m forms of the same degree is OF FIBER TYPE if the bihomogeneous defining ideal $\mathcal{J} \subset R[\mathbf{Y}] = R[Y_1, \ldots, Y_m]$ of the Rees algebra $\mathcal{R}(I)$ is generated by its $\mathbf{Y}$-linear forms and the defining equations of the special fiber $\mathcal{R}(I)/(\mathbf{X})\mathcal{R}(I)$.

Proposition 2.9. *Let* $\mathcal{L}$ *denote an* $m \times (m-1)$ *general linear matrix over* $R = k[X_1, \ldots, X_n]$, *with* $m \geq n \geq 3$. *Setting* $I := I_{m-1}(\mathcal{L}) \subset R$, *one has:*

(a) *The rational map* $\mathfrak{G} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{m-1}$ *defined by the* $(m-1)$*-minors is birational onto its image.*

(b) *I is an ideal of fiber type and the Rees algebra $\mathcal{R}(I)$ is a Cohen–Macaulay domain.*

(c) *The map* $\mathfrak{G}$ *admits* $\binom{m-1}{n-1}$ *source inversion factors, each associated to a minimal representative of the inverse map; moreover, they are all of degree* $(m-1)(n-1)-1$.

Proof. (a) By [8, Theorem 3.2], it suffices to prove that the dimension of the k-subalgebra of R generated by the minors has dimension n, i.e., that I has maximal analytic spread. The case where $m = n$ follows from Proposition 2.3 (iii). Now assume that $m > n$. Since R/I is Cohen–Macaulay and satisfies $\mu_P(I) \leq \operatorname{ht} P$, for $\operatorname{ht} P \leq n-1$ (Proposition 2.4 (i)), the result follows from [22, Theorem 4.3].

(b) For $m = n$ there is nothing to prove regarding the fiber type property, while the symmetric algebra is even a complete intersection. Thus, assume that $m > n$. In this case the result follows from [16, Theorem 1.3]. In addition, the defining ideal of the Rees algebra $\mathcal{R}(I)$ is $(I_1(\mathbf{X} \cdot B), I_n(B))$, where B denotes the Jacobian dual matrix of $\mathcal{L}$.

(c) Since I is of fiber type, a weak Jacobian dual matrix of I as in [8] coincides with the transpose B^t of the matrix introduced in the previous item; B^t is an $(m-1) \times n$ matrix of linear forms in the $\mathbf{Y}$-variables, whose rank over the special fiber of I is $n-1$. By part (a) and [8], any $(n-1) \times n$ submatrix has rank $n-1$ and its n (ordered, signed) maximal minors are the coordinates of a representative of the inverse map; thus, there are $\binom{m-1}{n-1}$ such representatives.

By construction, the degree of any one of these representatives (i.e., of its coordinates as elements of the special fiber) is exactly $n-1$. It follows from Proposition 1.3 that each such representative gives rise to a source inversion factor that is an element of the symbolic power $I^{(n-1)}$and has degree $(m-1)(n-1)-1$. □

One gets immediately the following preamble to the subsequent main results.

Proposition 2.10. *Let $\mathcal{L}$ denote an $m \times (m-1)$ general linear matrix over $R = k[X_1, \ldots, X_n]$, with $m \geq n \geq 3$. Set $I = I_{m-1}(\mathcal{L}) \subset R$. Then $I^{(r)} = I^r$ for $1 \leq r \leq n-2$, and $D_j \in I^{(n-1)} \setminus I^{n-1}$, where D_j ($j = 1, \ldots, \binom{m-1}{n-1}$) are the source inversion factors associated to a complete set of minimal representatives of the inverse map.*

Proof. The first assertion follows immediately from Theorem 2.8 and the second assertion stems from Proposition 1.3. □

Thus far, the available features of the theory are by nature general. In the subsequent part we come to grips with a richer amount of information, by focusing on the cases where $m = n$ or $m = n+1$. We will have to go a long way to obtain the nature of the corresponding symbolic Rees algebras. Structure theorems for $m \geq n+2$ are this far unknown (see Remark 2.24).

2.3 The symbolic algebra: Cremona case $m = n$

The classical theory of plane Cremona maps in characteristic zero relates the Jacobian of a homaloidal net with the principal curves of the corresponding Cremona map. Our first proposition for this part is a far-fetched analogue of this result.

Proposition 2.11. (char$(k) = 0$) *Let $R = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k of characteristic zero, with its standard grading and let $\mathcal{L} = (\ell_{ij})$ be an $n \times (n-1)$ matrix whose entries are linear forms in R. For every $i = 1, \ldots, n$ write Δ_i for the signed $(n-1)$-minor of $\mathcal{L}$ obtained omitting the i-th row and let $\Theta = \Theta(\mathbf{\Delta})$ denote the Jacobian matrix of $\mathbf{\Delta} := \{\Delta_1, \ldots, \Delta_n\}$.*

If the ideal $I_{n-1}(\mathcal{L}) := (\mathbf{\Delta}) \subset R$ is of linear type then the rational map $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ defined by $\mathbf{\Delta}$ is a Cremona map and the associated source inversion factor is $\frac{1}{n-1}\det(\Theta)$.

Proof. The first assertion to the effect that the map is birational is [18, Examples 2.4] (also [19, Theorem 3.12]).

We proceed to determine the source inversion factor. Consider the Jacobian dual matrix of [19] which is the Jacobian matrix with respect to $X_1, \ldots, X_n$ of the linear forms in the target variables $Y_1, \ldots, Y_n$ induced by the columns of $\mathcal{L}$. This is the following matrix:

$$\begin{pmatrix} \sum_{r=1}^n \frac{\partial \ell_{r1}}{\partial X_1} Y_r & \ldots & \sum_{r=1}^n \frac{\partial \ell_{r1}}{\partial X_n} Y_r \\ \vdots & & \vdots \\ \sum_{r=1}^n \frac{\partial \ell_{rn-1}}{\partial X_1} Y_r & \ldots & \sum_{r=1}^n \frac{\partial \ell_{rn-1}}{\partial X_n} Y_r \end{pmatrix}$$

Now, by [18, Examples 2.4] the inverse map is defined by the (signed) maximal minors of this matrix. Therefore, letting $\mathfrak{d}_i$ denote the signed $(n-1)$-minor of this matrix omitting the ith row, by definition of the source inversion factor we are to show that the outcome of evaluating $\mathfrak{d}_i$ via the map $Y_i \mapsto \Delta_i$ is $\frac{1}{n-1}\det(\Theta)\, X_i$.

To this purpose, we first note the following equality, where now Δ_i denotes the respective non-signed minor:

$$\sum_{r=1}^{n} (-1)^{n+r} \frac{\partial \ell_{rj}}{\partial X_k} \Delta_r = \sum (-1)^{n+r+1} \ell_{rj} \frac{\partial \Delta_r}{\partial X_k}, \text{ for } 1 \leq k \leq n,\ 1 \leq j \leq n-1,$$

from which we gather:

$$\mathfrak{d}_i(\boldsymbol{\Delta}) = \det \left(\sum_{r=1}^n (-1)^{n+r+1} \ell_{r1} \begin{pmatrix} \frac{\partial \Delta_r}{\partial X_1} \\ \vdots \\ \frac{\partial \Delta_r}{\partial X_{i-1}} \\ \frac{\partial \Delta_r}{\partial X_{i+1}} \\ \vdots \\ \frac{\partial \Delta_r}{\partial X_n} \end{pmatrix} \quad \ldots \quad \sum_{r=1}^n (-1)^{m+r+1} \ell_{rn-1} \begin{pmatrix} \frac{\partial \Delta_r}{\partial X_1} \\ \vdots \\ \frac{\partial \Delta_r}{\partial X_{r-1}} \\ \frac{\partial \Delta_i}{\partial X_{r+1}} \\ \vdots \\ \frac{\partial \Delta_r}{\partial X_n} \end{pmatrix} \right)$$

Write $[r_1 \ldots r_{n-1}]$ for the $(n-1)$-minor of $\mathcal{L}$ with rows $r_1, \ldots, r_{n-1}$ and let $\alpha_{r_1 \ldots r_{n-1}} := (n+1)(n-1) + \sum_{s=1}^n r_s$. By the multi-linearity of determinants, the result of evaluating $\mathfrak{d}_i$ is then

$$\sum_{1\le r_1<\ldots<r_{n-1}\le n} \left((-1)^{\alpha_{r_1\ldots r_{n-1}}}\sum_{\sigma}(-1)^{\sigma}\ell_{\sigma(r_1)1}\cdots\ell_{\sigma(r_{n-1})n-1}\right)\det\begin{pmatrix} \frac{\partial\Delta_{r_1}}{\partial X_1} & \cdots & \frac{\partial\Delta_{r_{n-1}}}{\partial X_1}\\ \vdots & \ddots & \vdots\\ \frac{\partial\Delta_{r_1}}{\partial X_{i-1}} & \cdots & \frac{\partial\Delta_{r_{n-1}}}{\partial X_{i-1}}\\ \frac{\partial\Delta_{r_1}}{\partial X_{i+1}} & \cdots & \frac{\partial\Delta_{r_{n-1}}}{\partial X_{i+1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial\Delta_{r_1}}{\partial X_n} & \cdots & \frac{\partial\Delta_{r_{n-1}}}{\partial X_n}\end{pmatrix}$$

$$= \sum_{1\le r_1<\ldots<r_{n-1}\le n}(-1)^{\alpha_{r_1\ldots r_{n-1}}}[r_1\ldots r_{n-1}]\det\begin{pmatrix} \frac{\partial\Delta_{r_1}}{\partial X_1} & \cdots & \frac{\partial\Delta_{r_{n-1}}}{\partial X_1}\\ \vdots & \ddots & \vdots\\ \frac{\partial\Delta_{r_1}}{\partial X_{i-1}} & \cdots & \frac{\partial\Delta_{r_{n-1}}}{\partial X_{i-1}}\\ \frac{\partial\Delta_{r_1}}{\partial X_{i+1}} & \cdots & \frac{\partial\Delta_{r_{n-1}}}{\partial X_{i+1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial\Delta_{r_1}}{\partial X_n} & \cdots & \frac{\partial\Delta_{r_{n-1}}}{\partial X_n}\end{pmatrix}$$

$$= \det\begin{pmatrix} \frac{\partial\Delta_1}{\partial X_1} & \cdots & \frac{\partial\Delta_n}{\partial X_1}\\ \vdots & \ddots & \vdots\\ \frac{\partial\Delta_1}{\partial X_{i-1}} & \cdots & \frac{\partial\Delta_n}{\partial X_{i-1}}\\ \Delta_1 & \cdots & \Delta_n\\ \frac{\partial\Delta_1}{\partial X_{i+1}} & \cdots & \frac{\partial\Delta_n}{\partial X_{i+1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial\Delta_1}{\partial X_n} & \cdots & \frac{\partial\Delta_n}{\partial X_n}\end{pmatrix} = \frac{X_i}{n-1}\det\Theta$$

where we have expanded the determinant by Laplace according to the ith row and used Euler's formula.

Corollary 2.12. ($\mathrm{char}(k)=0$) *Let $R=k[X_1,\ldots,X_n]$ be a polynomial ring over a field k of characteristic zero, with its standard grading and let $\mathcal{L}$ be an $n\times(n-1)$ general linear matrix. Then $I_{n-1}(\mathcal{L})$ is the base ideal of a Cremona map of $\mathbb{P}^{n-1}$ and the associated source inversion factor is $\frac{1}{n-1}\det(\Theta)$, where Θ denotes the Jacobian matrix of the $(n-1)$-minors of $\mathcal{L}$.*

Proof. The first assertion follows from Proposition 2.3. As for the second assertion, we observe that it can be deduced in two ways: one as a consequence of Proposition 2.11, while the other comes out of Proposition 1.2 by noticing that the inversion factor and $\det(\Theta)$ have the same degree, while the latter is an irreducible polynomial since the $(n-1)$-minors of $\mathcal{L}$ are sufficiently general forms. ☐

Here is the main theorem in the case $m=n$:

Theorem 2.13. *Let $\mathcal{L}$ denote an $n\times(n-1)$ general linear matrix over $R=k[X_1,\ldots,X_n]$, with $n\ge 3$. Set $I:=I_{n-1}(\mathcal{L})\subset R$ and let $\mathcal{R}^{(I)}$ denote its symbolic Rees algebra. Then*

(a) *$\mathcal{R}^{(I)}$ is a Gorenstein normal domain.*

(b) ($\mathrm{char}(k)=0$) *$\mathcal{R}^{(I)}$ is generated by the $(n-1)$-minors of $\mathcal{L}$, viewed in degree 1 and by the source inversion factor of the Cremona map defined by these minors, viewed in degree $n-1$.*

Moreover, this inversion factor coincides with a nonzero scalar multiple of the Jacobian determinant of the very minors.

Proof. (a) The symbolic Rees algebra $\mathcal{R}^{(I)}$ of I is a Gorenstein ring; indeed, it is a quasi-Gorenstein Krull domain since $\operatorname{ht} I = 2$ ([20]). On the other hand, by the proof of [21, Corollary 2.4 (b)], $\mathcal{R}^{(I)}$ is finitely generated since one has an isomorphism $\mathcal{R}^{(I)} \simeq \mathcal{R}(I)[t^{-1}] = R[It, t^{-1}]$. Moreover, the latter is Cohen–Macaulay since $\mathcal{R}(I)$ is Cohen–Macaulay by Proposition 2.9 (b). It follows that $\mathcal{R}^{(I)}$ is a Gorenstein normal domain.

(b) To get the explicit generation, let $\mathfrak{d}_1, \ldots, \mathfrak{d}_n \in k[\mathbf{Y}]$ be forms of the same degree, with gcd $= 1$, defining the inverse map and let $D \in R$ denote the corresponding source inversion factor. Write $J = (\mathfrak{d}_1, \ldots, \mathfrak{d}_n) \subset k[\mathbf{Y}]$. By definition, one has

$$D = \mathfrak{d}_i(\Delta_1, \ldots, \Delta_n)/X_i,\ 1 \leq i \leq n,$$

where $\mathbf{\Delta} := \{\Delta_1, \ldots, \Delta_n\}$ are the (signed) minors generating I. Identifying the two Rees algebras $\mathcal{R}_R(I) = R[It] \subset R[t]$ and $\mathcal{R}_{k[\mathbf{Y}]}(J) = k[\mathbf{Y}][Ju] \subset k[\mathbf{Y}][u]$ by a k-isomorphism that maps $Y_i \mapsto \Delta_i t$ and $X_i \mapsto \mathfrak{d}_i u$, then D is identified with $\mathfrak{d}_1/X_1$ in the common field of fractions. Drawing on Proposition 2.4 (ii) (here we need $\operatorname{char}(k) = 0$), then the symbolic algebra is generated by It and Dt^{n-1} as a consequence of [21, Corollary 2.4 (b)] (note that the notation for the two ideals is reversed in the latter).

The additional statement follows from Corollary 2.12 (again in characteristic zero). □

2.4 The symbolic algebra: implicitization case $m = n + 1$

We will now assume that $m = n + 1$.

2.4.1 Homological prelims

The arguments in this part will draw on the following results of independent interest. To describe their contents, recall that $\mathcal{S}_{n-1}(I) \simeq I^{n-1}$ by Corollary 2.5. Therefore, by [2, 23, 26] one has a free resolution of I^{n-1}

$$\mathcal{K}_{n-1}\ :\ 0 \to F_{n-1} \to F_{n-2} \to \ldots \to F_1 \to F_0 \to 0$$

where

$$F_i := \bigwedge^i R^n \otimes_R \mathcal{S}_{(n-1)-i}(R^{n+1})$$

and $d : F_i \to F_{i-1}$ is given by

$$d(e_1 \wedge \ldots \wedge e_i \otimes g) := \sum_{l=1}^{i} e_1 \wedge \ldots \wedge \widehat{e_l} \wedge \ldots \wedge e_i \otimes \varphi(e_l) g,$$

with $\{e_1, \ldots, e_n\}$ denoting a basis of R^n and $\varphi : F_1 \simeq R^n \to F_0 \simeq R^{n+1}$ standing for the map defined by the presentation matrix $\mathcal{L} = (\ell_{ij})_{\substack{1 \leq i \leq n+1 \\ 1 \leq j \leq n}}$ of the ideal I.

Consider the R-dual map to $d_{n-1} : F_{n-1} \to F_{n-2}$. Since I^{n-1} is generated in (standard) degree $n(n-1)$, after identification and taking in account the degrees shift, the dual map is of the form

$$\eta := d_{n-1}^{*} : R^N((n+1)(n-1) - 1) \to R^n((n+1)(n-1)), \tag{9}$$

where $N = (n+1)\binom{n}{2}$. Let M denote the cokernel of η. Shifting by $-((n+1)(n-1))$, we get a homogeneous presentation

$$R^N(-1) \overset{\eta}{\to} R^n \to M(-(n+1)(n-1)) \to 0.$$

Theorem 2.14. *With the above notation, there is a homogeneous isomorphism*

$$M(-(n+1)(n-1)) \simeq R^n/(\mathbf{X})R^n = k^n.$$

Proof. Picking up from the above preliminaries, let us make explicit the dual map to $d_{n-1} : F_{n-1} \to F_{n-2}$. Note that

$$F_{n-1} = \bigwedge^{n-1} R^n \otimes_R S_0(R^{n+1}) \simeq R^n, \quad F_{n-2} = \bigwedge^{n-2} R^n \otimes_R S_1(R^{n+1}) \simeq R^{\binom{n}{n-2}} \otimes_R R^{n+1}.$$

Applying these identifications, the basis vector $e_1 \wedge \cdots \wedge \widehat{e_k} \wedge \cdots \wedge e_n$ gets identified with e_k and we write $a_{1,\ldots,\hat{k},\ldots,\hat{l},\ldots,n}$ for a basis vector of $R^{\binom{n}{n-2}}$ corresponding to $e_1 \wedge \cdots \wedge \widehat{e_j} \wedge \cdots \wedge \widehat{e_l} \wedge \cdots \wedge e_n$. Further, let $\{b_1, \ldots, b_{n+1}\}$ stand for a basis of R^{n+1} With this notation, for $k = 1, \ldots, n$, the map is quite simply

$$\begin{aligned} e_k \quad \mapsto \quad & \sum_{l=1}^{n-1} a_{1,\ldots,\hat{k},\ldots,\hat{l},\ldots,n} \otimes \varphi(e_l) = \sum_{l=1}^{n-1} a_{1,\ldots,\hat{k},\ldots,\hat{l},\ldots,n} \otimes \sum_{i=1}^{n+1} \ell_{il} b_i \\ = \quad & \sum_{i=1}^{n+1} \sum_{l=1}^{n-1} \ell_{il}\, a_{1,\ldots,\hat{k},\ldots,\hat{l},\ldots,n} \otimes b_i, \end{aligned}$$

where $\mathcal{L} = (\ell_{ij})_{\substack{1 \le i \le n+1 \\ 1 \le j \le n}}$ stands for the structural matrix of the ideal I.

From this the transposed matrix has the following block shape

$$\eta = (M_{n-1,n}| \ldots |M_{1,n}| \ldots |M_{j-1,j}| \ldots |M_{1,j}| \ldots |M_{1,2}),$$

where, for $1 \le i \le j \le n$, M_{ij} is the following $n \times (n+1)$ matrix up to signs

$$M_{i,j} = \begin{array}{cl} \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ \ell_{1i} & \ell_{2i} & \ldots & \ell_{(n+1)i} \\ \vdots & \vdots & \ldots & \vdots \\ \ell_{1j} & \ell_{2j} & \ldots & \ell_{(n+1)j} \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} & \begin{array}{l} \\ \\ \leftarrow (n+1-j)\text{th row} \\ \\ \leftarrow (n+1-i)\text{th row} \\ \\ \\ \end{array} \end{array}$$

Next let $\widetilde{M_{i,j}}$ denote the submatrix of $M_{i,j}$ consisting of the first n columns and consider the following block submatrix of η

$$(\widetilde{M_{n-1,n}}| \ldots |\widetilde{M_{1,n}} \,|\, \widetilde{M_{n-2,n-1}}),$$

consisting of n square blocks of order n each; in particular, the matrix has n^2 columns.

We claim that the R-submodule of R^n generated by the columns of the above matrix coincides with $(\mathbf{X})R^n$. For this, since the columns have standard degree 1, it suffices to show that the columns are k-linearly independent as elements of the k-vector space $((\mathbf{X})R)_1$.

Suppose that a nontrivial k-linear combination of these columns vanishes, with coefficients $\alpha_1, \ldots, \alpha_{n^2} \in k$. Grouping the coefficients corresponding to the variables $X_1, \ldots, X_n$ one gets an $n \times n$ linear system with coefficients in k such that $\{X_1, \ldots, X_n\}$ is a non zero solution. But then every row of the system gives a k-linear relation of these variables. Clearly this is only possible if all the coefficients of this system vanish. Writing this condition as a new square linear system, this time around of order n^2 with solution $\{\alpha_1, \ldots, \alpha_{n^2}\}$ and appropriate coefficients in k. Since the latter coefficients are nothing but the coefficients of all linear forms ℓ_{ij}, they can be expressed as partial derivatives of these forms, so the corresponding $n^2 \times n^2$ matrix has the following form (up to signs)

$$\Theta = \begin{pmatrix} \Theta_{n-1} & \Theta_{n-2} & \Theta_{n-3} & \ldots & \Theta_2 & \Theta_1 & \mathbf{0} \\ \Theta_n & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \Theta_{n-2} \\ \mathbf{0} & \Theta_n & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \Theta_{n-1} \\ \mathbf{0} & \mathbf{0} & \Theta_n & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \Theta_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \Theta_n & \mathbf{0} \end{pmatrix}$$

where Θ_i is the transpose of the Jacobian matrix of $(\ell_{1,i}, \ldots, \ell_{n,i})$ and $\mathbf{0}$ denotes the null matrix of order n. The system has only the trivial solution if and only if the determinant of this matrix does not vanish. To see this non-vanishing we use the full force of the assumption that the entries of $\mathcal{L}$ are general linear forms. This implies that, up to a projective change of coordinates, we may assume that $\ell_{in} = X_i$, for $1 \leq i \leq n$. But then Θ_i is the identity matrix of order n and the matrix takes the form

$$\Theta = \begin{pmatrix} \Theta_{n-1} & \Theta_{n-2} & \Theta_{n-3} & \ldots & \Theta_2 & \Theta_1 & \mathbf{0} \\ I & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \Theta_{n-2} \\ \mathbf{0} & I & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \Theta_{n-1} \\ \mathbf{0} & \mathbf{0} & I & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & I & \mathbf{0} \end{pmatrix}$$

One can see that, after appropriate elementary row operations, the above determinant is non-vanishing if and only if the determinant of the following matrix does not vanish:

$$\begin{pmatrix} I & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \Theta_{n-2} \\ \mathbf{0} & I & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \Theta_{n-1} \\ \mathbf{0} & \mathbf{0} & I & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & 0 & \Omega \end{pmatrix}$$

where $\Omega = \Theta_{n-1}\Theta_{n-2} - \Theta_{n-2}\Theta_{n-1}$. Thus, $\det(\Theta) \neq 0$ if and only if $\det(\Omega) \neq 0$. Again, since the entries of $\mathcal{L}$ are general linear forms, the previous change of variables does not affect the forms, other than $\ell_{in}, 1 \leq i \leq n$, in their nature of general linear forms. Therefore, one must have $\det(\Omega) \neq 0$.

Now, to conclude, we have shown that the image of the map η in (9) is the R-submodule $(\mathbf{X})R^n$. Therefore, $M(-(n+1)(n-1)) \simeq R^n/(\mathbf{X})R^n$ as required. □

Example 2.15. The above discussion has many common points with [24, Section 8.2] which treats the case of linearly presented perfect ideals in dimension $n = 3$. However, the above proof draws heavily on the hypothesis that the entries of the matrix $\mathcal{L}$ are general linear forms – and, in fact, it may be false for other linearly presented ideals. We are indebted to A. Tchernev for having provided us with the following counter-example to Theorem 2.14 in the context of arbitrary linearly presented ideals:

$$\varphi = \begin{pmatrix} X_1 & X_2 & X_3 \\ X_2 & X_3 & 0 \\ X_3 & 0 & X_1 \\ 0 & X_1 & X_2 \end{pmatrix}$$

Here the vector space dimension of the linear forms in $\mathrm{Im}(\eta)$ is 8, where η denotes the correspondingly defined matrix as in the proposition.

Proposition 2.16. *Let $\mathcal{L}$ denote an $(n+1) \times n$ general linear matrix over $R = k[X_1, \ldots, X_n]$, with $n \geq 3$. Set $I = I_n(\mathcal{L}) \subset R$. Then*

$$I^{(n-1)}/I^{n-1} \simeq k^n(-(n(n-1)-1)),$$

as graded R-modules.

Proof. By Theorem 2.14, one has a (shifted) homogeneous isomorphism

$$M(-n) \simeq k^n(n(n-1)-1).$$

On the other hand, by definition there is a homogeneous isomorphism

$$M \simeq \mathrm{Ext}^n_R(R/I^{n-1}, R).$$

Therefore, it obtains

$$\begin{aligned} I^{(n-1)}/I^{n-1} &\simeq H^0_{(\mathbf{X})}(R/I^{n-1}) \quad (\text{since } I^{(n-1)}/I^{n-1} \text{ has finite length}) \\ &\simeq \mathrm{Hom}_R(\mathrm{Ext}^n_R(R/I^{n-1}, R(-n)), E(k)) \quad (\text{by graded local duality}) \\ &\simeq \mathrm{Hom}_R(M(-n), E(k)) \simeq \mathrm{Hom}_R(k^n(n(n-1)-1), E(k)) \\ &\simeq \mathrm{Hom}_R(k, E(k))^{\oplus n}(-(n(n-1)-1)) \simeq k^n(-(n(n-1)-1)), \end{aligned}$$

where the last isomorphism is given in [4, Lemma 3.2.7 (b)]. □

Example 2.17. Corollary 2.16 fails for arbitrary perfect ideals of codimension 2 admitting linear presentation. For $n = 3$, Example 2.15 is a counter-example. Letting $I \subset R = k[X_1, X_2, X_3]$ denote the ideal of 3-minors, then $I^{(2)}/I^2$ is a cyclic R-module generated by the residue class of

a form $F \in I^{(2)}$ of degree $4 < n(n-1)-1 = 5$. The map defined by the minors is still birational onto the image, with inversion factors X_1F, X_2F, X_3F. In particular, the latter are not minimal generators of $I^{(2)}$. Even if we slightly "perturb" Tchernev's matrix the result equally fails, such as in the following matrix

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ X_2 & X_3 & 0 \\ X_3 & 0 & X_1 - X_2 \\ 0 & X_1 - X_3 & X_2 - X_3 \end{pmatrix}.$$

In both examples we are careful to make sure that I is a radical ideal, otherwise the whole known repository of symbolic power theory crumbles down. Therefore, slightly perturbing a linear $(n+1) \times n$ matrix whose n-minors generate a radical ideal may lead us astray. As an example, changing the lower right corner entry of the above matrix into $X_1 - X_2 + X_3$ gives a non-radical ideal.

2.4.2 The trick of the transposed Jacobian dual

A good deal of the subsequent development rests on a simple construction.

Namely, let $\mathbf{\Delta} = \{\Delta, \ldots, \Delta_{n+1}\}$ denote the signed maximal minors of $\mathcal{L}$. Let B denote the Jacobian dual matrix of $\mathcal{L}$, whose entries belong to the polynomial ring $k[\mathbf{Y}] = k[Y_1, \ldots, Y_n, Y_{n+1}]$. By definition, one has an equality $\mathbf{Y} \cdot \mathcal{L} = \mathbf{X} \cdot B^t$, where the superscript ^{t} denotes transpose. We can similarly write an equality $\mathbf{Y} \cdot \mathcal{L}' = \mathbf{Z} \cdot B$, for a unique matrix $\mathcal{L}'$ whose entries are linear forms in a set of duplicate variables $\mathbf{Z}$ of $\mathbf{X}$.

We observe that $\mathcal{L}'$ only differs from $\mathcal{L}$ by the rearrangement of the (same) coefficients of the linear forms; it follows that it too has entries which are general linear forms in $\mathbf{Z}$. Therefore, its n-minors $\boldsymbol{\delta} = \{\delta_1, \ldots, \delta_{n+1}\}$ define a birational map onto the image, with B^t as its Jacobian dual matrix and corresponding set $\{d_1(\mathbf{Z}), \ldots, d_n(\mathbf{Z})\}$ of source inversion factors associated to a complete set of minimal representatives of the corresponding inverse map.

As usual, when we list the set of n-minors we normally mean the signed such minors. Keeping the above notation, one has the following basic structural result:

Theorem 2.18. *Let $\mathcal{L}$ denote an $(n+1) \times n$ $(n \geq 3)$ general linear matrix over $R = k[\mathbf{X}] = k[X_1, \ldots, X_n]$ with n-minors $\mathbf{\Delta} = \{\Delta_1, \ldots, \Delta_{n+1}\}$ and let $\{D_1, \ldots, D_n\}$ and $\{d_1(\mathbf{Z}), \ldots, d_n(\mathbf{Z})\}$ be as above. Then:*

(i) *$\{D_1, \ldots, D_n\} \subset R$ and $\{d_1(\mathbf{Z}), \ldots, d_n(\mathbf{Z})\} \subset k[\mathbf{Z}]$ both generate ideals of codimension 2.*

(ii) *$\{D_1, \ldots, D_n\}$ defines a Cremona map $\mathfrak{D}$ of $\mathbb{P}^{n-1}$ whose inverse map is $(d_1(\mathbf{Z}) : \cdots : d_n(\mathbf{Z}))$.*

(iii) *Writing $I := (\Delta_1, \ldots, \Delta_{n+1})$, the R-module $I^{(n-1)}/I^{n-1}$ is minimally generated by the classes of $D_1, \ldots, D_n$; in particular, the symbolic power $I^{(n-1)}$ is minimally generated by $D_1, \ldots, D_n$ and by the minimal generators of I^{n-1} which are not of the form $X_iD_j, 1 \leq i, j \leq n$.*

(iv) *The source inversion factor of $\mathfrak{D}$ is the $(n-1)$th power of an element $E \in I^{(n(n-1)-1)}$.*

Supplement*: if, moreover,* $\mathrm{char}(k) = 0$ *then E coincides with the Jacobian determinant of $D_1, \ldots, D_n$.*

(v) *The minimal graded resolution of the ideal* $(D_1,\ldots,D_n)\subset R$ *is*

$$0\to\ R(-n^2)\ \stackrel{\mathbf{X}^t}{\to}\ R(-(n^2-1))^n\ \stackrel{\Psi}{\to} R(-(n(n-1)-1))^n\ \to R, \tag{10}$$

where Ψ *denotes the Jacobian dual matrix of the signed* n*-minors of* $\mathcal{L}$ *evaluated orderly on these signed minors, while* $\mathbf{X}^t$ *stands for the transpose of the vector of the source variables.*

Proof. (i) We only discuss the ideal $(D_1,\ldots,D_n)$ since the line of argument is analogous for $(d_1(\mathbf{Z}),\ldots,d_n(\mathbf{Z}))$.

Being a subideal of $I:=(\Delta_1,\ldots,\Delta_{n+1})$, the codimension of $(D_1,\ldots,D_n)$ is at most 2. Thus, it suffices to show that it is at least 2. Start from scratch by observing that $k[\mathbf{\Delta}]\simeq k[\mathbf{Y}]/(\boldsymbol{\beta})$, where $\boldsymbol{\beta}:=\det(B)$ and B stands for the Jacobian dual matrix of $\mathbf{\Delta}$. Since $\boldsymbol{\beta}(\mathbf{\Delta})=0$, the chain rule of derivatives gives the short polarization complex

$$R\stackrel{\partial}{\longrightarrow}R^{n+1}\stackrel{\Theta}{\longrightarrow}R^n, \tag{11}$$

where Θ denotes the transposed Jacobian matrix of $\mathbf{\Delta}$ and ∂ is the transpose of

$$\left[\frac{\partial\boldsymbol{\beta}}{\partial Y_1}(\mathbf{\Delta})\ \cdots\ \frac{\partial\boldsymbol{\beta}}{\partial Y_{n+1}}(\mathbf{\Delta})\right].$$

On the other hand, since $\dim k[\mathbf{\Delta}]=n$, the rank of Θ is n, hence $\ker(\Theta)$ is generated by the single (column) vector whose jth coordinate is the n-minor of Θ omitting the jth column of Θ further divided by the gcd of all the n-minors. Since $\mathbf{\Delta}$ are maximal minors of a general linear matrix, they are sufficiently general n-forms, and so are any of their derivatives (the entries of Θ), and hence the ideal $I_n(\Theta)$ generated by the maximal minors of Θ has codimension 2. This implies that $\ker(\Theta)$ is generated by a single vector in degree $(n-1)n$ (the degree of an n-minor of Θ). On the other hand, a simple calculation shows that the coordinates of ∂ are also of degree $n(n-1)$. Since by (11) the jth coordinate of ∂ is a multiple of the n-minor of Θ omitting the jth column, we must conclude that the ideals $\left(\frac{\partial\boldsymbol{\beta}}{\partial Y_1}(\mathbf{\Delta})\ \ldots\ \frac{\partial\boldsymbol{\beta}}{\partial Y_{n+1}}(\mathbf{\Delta})\right)$ and $I_n(\Theta)$ coincide. In particular, the first of these ideals has codimension 2. We proceed to show that it is further contained in the ideal $(D_1,\ldots,D_n)$, thus showing that the latter has codimension at least 2.

Let $\mathcal{L}=(\ell_{ij})$ denote the given general linear $(n+1)\times n$ matrix. Then

$$B^t=\begin{pmatrix} \sum_{r=1}^{n+1}\frac{\partial\ell_{r,1}}{\partial X_1}Y_r & \sum_{r=1}^{n+1}\frac{\partial\ell_{r,2}}{\partial X_1}Y_r & \cdots & \sum_{r=1}^{n+1}\frac{\partial\ell_{r,n}}{\partial X_1}Y_r\\ \sum_{r=1}^{n+1}\frac{\partial\ell_{r,1}}{\partial X_2}Y_r & \sum_{r=1}^{n+1}\frac{\partial\ell_{r,2}}{\partial X_2}Y_r & \cdots & \sum_{r=1}^{n+1}\frac{\partial\ell_{r,n}}{\partial X_2}Y_r\\ \vdots & \vdots & \ddots & \vdots\\ \sum_{r=1}^{n+1}\frac{\partial\ell_{r,1}}{\partial X_n}Y_r & \sum_{r=1}^{n+1}\frac{\partial\ell_{r,2}}{\partial X_n}Y_r & \cdots & \sum_{r=1}^{n+1}\frac{\partial\ell_{r,n}}{\partial X_n}Y_r \end{pmatrix}.$$

Expanding the determinant of B, it obtains (up to signs)

$$\boldsymbol{\beta}=\sum_{1\le j_1\le\cdots\le j_n\le n}\left[\left(\sum_{r=1}^{n+1}\frac{\partial\ell_{r,j_1}}{\partial X_1}Y_r\right)\cdots\left(\sum_{r=1}^{n+1}\frac{\partial\ell_{r,j_n}}{\partial X_1}Y_r\right)\right].$$

Taking the kth derivative yields

$$\begin{aligned}\frac{\partial \boldsymbol{\beta}}{\partial X_k} &= \sum_{1\le j_1\le \cdots \le j_n\le n}\left[\sum_{1\le s\le n}\left(\sum_{r=1}^{n+1}\frac{\partial \ell_{r,j_1}}{\partial X_1}Y_r\right)\cdots\left(\frac{\partial \ell_{k,j_s}}{\partial X_s}\right)\cdots\left(\sum_{r=1}^{n+1}\frac{\partial \ell_{r,j_n}}{\partial X_1}Y_r\right)\right]\\ &= \sum_{1\le s\le n}\left[\sum_{1\le j_1\le \cdots \le j_n\le n}\left(\sum_{r=1}^{n+1}\frac{\partial \ell_{r,j_1}}{\partial X_1}Y_r\right)\cdots\left(\frac{\partial \ell_{k,j_s}}{\partial X_s}\right)\cdots\left(\sum_{r=1}^{n+1}\frac{\partial \ell_{r,j_n}}{\partial X_1}Y_r\right)\right]\end{aligned}$$

Note that for any given $1\le s\le n$, the expression inside the square brackets in the last line above is (up to signs) the determinant of the matrix

$$B_s := \begin{pmatrix} \sum_{r=1}^{n+1}\frac{\partial \ell_{r,1}}{\partial X_1}Y_r & \sum_{r=1}^{n+1}\frac{\partial \ell_{r,2}}{\partial X_1}Y_r & \ldots & \sum_{r=1}^{n+1}\frac{\partial \ell_{r,n}}{\partial X_1}Y_r\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial \ell_{k,1}}{\partial X_s} & \frac{\partial \ell_{k,2}}{\partial X_s} & \ldots & \frac{\partial \ell_{k,n}}{\partial X_s}\\ \vdots & \vdots & \ddots & \vdots\\ \sum_{r=1}^{n+1}\frac{\partial \ell_{r,1}}{\partial X_n}Y_r & \sum_{r=1}^{n+1}\frac{\partial \ell_{r,2}}{\partial X_n}Y_r & \ldots & \sum_{r=1}^{n+1}\frac{\partial \ell_{r,n}}{\partial X_n}Y_r \end{pmatrix}.$$

Expanding this determinant one again, this tiem around by Laplace along the ith row of B_s, gives $\det(B_s)=\sum_{t=1}^{n}\frac{\partial \ell_{k,t}}{\partial X_s}\,\sigma_t^{[s]}$, where $\sigma_t^{[s]}$ denotes the $(n-1)$-minor of B_s omitting the sth row and the tth column. Coming from the other end, for given s, $(\sigma_1^{[s]}:\cdots:\sigma_n^{[s]})$ is a representative of the inverse map to the map defined by $\boldsymbol{\Delta}$ ([8, Theorem 2.18, Supplement]). By definition, say, D_s is the source inversion factor corresponding to this representative, hence $\sigma_t^{[s]}(\boldsymbol{\Delta}) = X_tD_s$, for $1\le s,t\le n$. Assembling the information, we get

$$\begin{aligned}\frac{\partial \boldsymbol{\beta}}{\partial X_k}(\boldsymbol{\Delta}) &= \det(B_1)(\boldsymbol{\Delta})+\cdots+\det(B_n)(\boldsymbol{\Delta}) = \sum_{t=1}^{n}\frac{\partial \ell_{k,t}}{\partial X_1}\sigma_t^{[1]}(\boldsymbol{\Delta})+\cdots+\sum_{t=1}^{n}\frac{\partial \ell_{k,t}}{\partial X_n}\sigma_t^{[n]}(\boldsymbol{\Delta})\\ &= \left(\sum_{t=1}^{n}\frac{\partial \ell_{k,t}}{\partial X_1}X_t\right)D_1+\cdots+\left(\sum_{t=1}^{n}\frac{\partial \ell_{k,t}}{\partial X_n}X_t\right)D_n,\end{aligned}$$

which proves our contention.

(ii) Let $\boldsymbol{\delta}=\{\delta_1,\ldots,\delta_{n+1}\}\subset k[\mathbf{Z}]$ stand for the n-minors of the general linear matrix $\mathcal{L}'$ as explained above. We have seen in the preliminaries of this section that they define a birational map onto the image, with B^t as its Jacobian dual matrix. Thus, for any $j\in\{1,\ldots,n\}$, the coordinate forms

$$(B^t_{j1},\ldots,B^t_{jn})$$

taken modulo $\det(B)$ define an inverse to the map defined by $\boldsymbol{\delta}$, thus yielding the following structural congruencies

$$\delta_i(B^t_{j1},\ldots,B^t_{jn})\equiv E_jY_i \mod (\det B^t), \tag{12}$$

where B^t_{ij} stands for the (i,j)-cofactor of B^t and E_j's are the target inversion factors.

Claim. $I_1(\mathbf{X}\cdot B^t(\boldsymbol{\delta}))$ is contained in the presentation ideal of the Rees algebra of the ideal $(D_1,\ldots,D_n)\subset k[\mathbf{X}]$, defined over the ring $k[\mathbf{X},\mathbf{Z}]$.

To see this it suffices to prove that the entries of $\mathbf{X}\cdot B^t$ vanish by evaluating $Y_k \mapsto \delta_k(D_1,\ldots,D_n)$, $k=1,\ldots,n+1$, or, equivalently, by evaluating $Y_k \mapsto \delta_k(X_nD_1,\ldots,X_nD_n)$, $k=1,\ldots,n+1$. Letting, as previously, $\boldsymbol{\Delta}=\{\Delta_1,\ldots,\Delta_{n+1}\}$ denote the signed n-minors of $\mathcal{L}$, one has the relations

$$X_nD_i = B_{in}(\Delta_1,\ldots,\Delta_{n+1}) = B^t_{ni}(\Delta_1,\ldots,\Delta_{n+1}), \tag{13}$$

since D_i is inversion factor for $\boldsymbol{\Delta}$, where B_{ij} is the cofactor of B corresponding to the entry indexed by (i,j) and $B_{in}(\Delta_1,\ldots,\Delta_{n+1})$ is the result of evaluating this cofactor on $\boldsymbol{\Delta}$. Since $B_{ij}=B^t_{ji}$, one gets

$$\begin{aligned}
\mathbf{X}\cdot B^t(\delta(X_nD_1,\ldots,X_nD_n)) &= \mathbf{X}\cdot\begin{pmatrix} \sum_{i=1}^{n+1}\frac{\partial\ell_{i1}}{\partial X_1}\delta_i(X_n\mathbf{D}) & \ldots & \sum_{i=1}^{n+1}\frac{\partial\ell_{in}}{\partial X_1}\delta_i(X_n\mathbf{D})\\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n+1}\frac{\partial\ell_{i1}}{\partial X_3}\delta_i(X_n\mathbf{D}) & \ldots & \sum_{i=1}^{n+1}\frac{\partial\ell_{in}}{\partial X_n}\delta_i(X_n\mathbf{D})\end{pmatrix} && (14)\\
&= \left(\sum_{i=1}^{n+1}\ell_{i1}\delta_i(X_n\mathbf{D}),\ldots,\sum_{i=1}^{n+1}\ell_{in}\delta_i(X_n\mathbf{D})\right) && (15)\\
&= E_n(\boldsymbol{\Delta})\left(\sum\ell_{i1}\Delta_i,\ldots,\sum\ell_{in}\Delta_i\right) && (16)\\
&= (0,\ldots,0) && (17)
\end{aligned}$$

where equality (16) follows from (12), (13) and (15) – keep in mind that the result of evaluating $\det(B^t)$ by $Y_K\mapsto\Delta_k$ is zero. As to equality (17), it is a consequence of (16) using that $\mathcal{L}$ is a syzygy matrix of $\boldsymbol{\Delta}$. This proves the claim.

As a consequence, the matrix $B(\boldsymbol{\delta})$ is a submatrix of the full Jacobian dual matrix of $\mathbf{D}:=\{D_1,\ldots,D_n\}$. On the other hand, we have $\det(B(\boldsymbol{\delta}))=(\det(B))(\boldsymbol{\delta})=(\det(B^t))(\boldsymbol{\delta})=0$ since $(\det(B^t))$ is a polynomial relation of $\boldsymbol{\delta}$. Therefore, $B(\boldsymbol{\delta})$ has rank $\leq n-1$. But since $\boldsymbol{\delta}$ defines a birational map, not all $(n-1)$-minors vanish modulo $\det(B^t)$. Thus, $B(\boldsymbol{\delta})$ has rank $n-1$. For even more reason, the rank of the Jacobian dual matrix of $\mathbf{D}$ is $\geq n-1$ (hence $=n-1$, its maximal possible value). Using again the criterion of [8] we derive that $\mathbf{D}$ defines a Cremona map.

Now, we prove the additional statement of this item. Let s denote the minimal number of generators of the Rees ideal of $\mathbf{D}$ of bidegree $(1,*)$, with $*$ representing any value ≥ 1. Then the full Jacobian dual matrix of $\mathbf{D}$ is an $s\times n$ matrix over $k[\mathbf{Y}]$ of rank $n-1$ which, as we have just shown, contains the $n\times n$ submatrix $B(\boldsymbol{\delta})$. By [8, Theorem 2.18, Supplement] we know that the inverse map to the Cremona map defined by $\mathbf{D}$ takes as its coordinate functions the $(n-1)$-minors of any $(n-1)\times n$ submatrix of rank $n-1$ of the Jacobian dual matrix of $\mathbf{D}$, further divided by their gcd. Since $B(\boldsymbol{\delta})$ has rank $n-1$, one can take, say, the submatrix of $B(\boldsymbol{\delta})$ formed with the first $n-1$ rows of $B(\boldsymbol{\delta})$. Write $\partial_i(\mathbf{Z})$ for the $(n-1)$th minor omitting the ith column. Then we get $\partial_i(\mathbf{Z})=B_{ni}(\boldsymbol{\delta})=B^t_{ni}(\boldsymbol{\delta})=X_nd_i(\mathbf{Z})$, where $d_i(\mathbf{Z})$ as before denotes the corresponding source inversion factor of the birational map defined by $\boldsymbol{\delta}$. It follows that $(d_1(\mathbf{Z}):\cdots:d_n(\mathbf{Z}))$ defines the inverse map to $\mathfrak{D}$.

(iii) By Proposition 2.9, one has $(I^{n-1},D_1,\ldots,D_n)\subset I^{(n-1)}$. On the other hand, by Proposition 2.16, $I^{(n-1)}/I^{n-1}$ is minimally generated by n elements of degree $n(n-1)-1$. To conclude that the residues of $D_1,\ldots,D_n$ on $I^{(n-1)}/I^{n-1}$ form a set of minimal generators of the latter it suffices to show that they are k-linearly independent. By part (i) they are even k-algebraically independent.

(iv) By (i) and (ii), $\{d_1 = d_1(\mathbf{Z}), \ldots, d_n = d_n(\mathbf{Z})\}$ generate an ideal of codimension 2 defining the inverse map to $\mathfrak{D}$. Write

$$h_i = h_i(Z_1, \ldots, Z_n) := Z_i d_i \; (= B^t_{ii}(\boldsymbol{\delta})), i = 1, \ldots, n$$

Evaluate h_i on $X_i\mathbf{D} = (X_i D_1, \ldots, X_i D_n)$ (i.e., through $Z_j \mapsto X_i D_j$):

$$\begin{aligned} h_i(X_i D_1, \ldots, X_i D_n) &= X_i D_i d_i(X_i D_1, \ldots, X_i D_n) \\ &= X_i^{n(n-1)} D_i d_i(D_1, \ldots, D_n) \\ &= X_i^{n(n-1)+1} D_i G \end{aligned}$$

where $G := X_i^{-1}\, d_i(D_1, \ldots, D_n)$ is the source inversion factor of the Cremona map defined by $\mathbf{D}$.

On the other hand, one has

$$\begin{aligned} h_i(X_i D_1, \ldots, X_i D_n) &= B^t_{i,i}(\delta_1(X_i\mathbf{D}), \ldots, \delta_n(X_i\mathbf{D})) \\ &= B^t_{i,i}(E_i(\boldsymbol{\Delta})\Delta_1, \ldots, E_i(\boldsymbol{\Delta})\Delta_{n+1}) \\ &= E_i(\boldsymbol{\Delta})^{n-1} B^t_{i,i}(\Delta_1, \ldots, \Delta_n) \\ &= E_i(\boldsymbol{\Delta})^{n-1} X_i D_i, \end{aligned}$$

where E_i, $i = 1, \ldots, n$ are a complete set of target inversion factors of the birational map defined by $\boldsymbol{\delta}$, as in (12). This implies the relation

$$X_i^{n(n-1)} G = E_i(\boldsymbol{\Delta})^{n-1}. \tag{18}$$

Extracting $(n-1)$th roots yields

$$X_i^n G^{1/n-1} = E_i(\boldsymbol{\Delta}) \tag{19}$$

Since E_i has degree $n(n-1) - 1$ then $(X_1^n, \ldots, X_n^n) G^{1/n-1} \subset I^{n(n-1)-1}$, from which follows $E := G^{1/n-1} \in I^{(n(n-1)-1)}$.

The supplementary statement follows from Proposition 1.2 since $\det(\Theta(\mathbf{D})$ is an irreducible polynomial. Indeed, each D_i is an inversion factor of a Cremona map whose defining coordinates $\boldsymbol{\Delta}$ are sufficiently general forms; as such it too is a sufficiently general polynomial ("general contracted divisor"), and so are their partial derivatives. It follows that $\det(\Theta(\mathbf{D})$ is an irreducible polynomial; as it divides a power of E it divides E as well, and since $\deg(\det(\Theta(\mathbf{D}))) = \deg(E)$, they coincide up to a nonzero scalar.

(v) We first check that (10) is indeed a complex. For this, using that $\{D_1, \ldots, D_n\}$ is a complete set of inversion factors of the birational map defined by $\boldsymbol{\Delta}$, the cofactor matrix of Ψ is

$$\mathrm{adj}(\Psi) = \begin{pmatrix} X_1 D_1 & X_1 D_2 & \ldots & X_1 D_n \\ X_2 D_1 & X_2 D_2 & \ldots & X_2 D_n \\ \vdots & \vdots & \ldots & \vdots \\ X_n D_1 & X_n D_2 & \ldots & X_n D_n \end{pmatrix} \tag{20}$$

Since Ψ has rank $n-1$, the cofactor equation gives

$$\mathrm{adj}(\Psi) \cdot \Psi = \mathbf{0} \tag{21}$$

and

$$\Psi \cdot \mathrm{adj}(\Psi) = \mathbf{0} \tag{22}$$

From (20), (21) implies that Ψ is a matrix of syzygies of $\mathbf{D}$, while (22) gives that $\mathbf{X}^t$ is a second syzygy thereof. This shows that one has indeed a complex. To finish we check the required Fitting codimension by the Buchsbaum–Eisenbud acyclicity criterion. The verification at the tail of the complex is immediate, while at the middle the codimension of

$$I_{n-1}(\Psi) = I_1(\mathrm{adj}(\Psi)) = (\mathbf{X})(D_1, \ldots, D_n)$$

is 2 because (i) showed that the ideal $(D_1, \ldots, D_n)$ has codimension 2. □

Remark 2.19. Assertion (i) in the last theorem depends once more on the general linear assumption; thus, in Example 2.17 the polarization complex is not exact and, in fact, $\{D_1, D_2, D_3\}$ admit a proper common factor.

2.4.3 The structure of the symbolic algebra

Here is the degree numerology so far:

- $\deg(d_i) = \deg(D_i) = n(n-1) - 1$, for $i = 1, \ldots, n$
- $\deg(G) = (n(n-1)-1)n(n-1) - n(n-1) = (n-1)n(n(n-1)-2)$ – from (18).
- $\deg(E) = \deg(G)/(n-1) = n(n(n-1)-2)$.

Further consideration is given in the following strategic lemma:

Lemma 2.20. *Let $\mathcal{L}$ denote an $(n+1) \times n$ general linear matrix over $R = k[X_1, \ldots, X_n]$, with $n \geq 3$. Set $I := I_{n-1}(\mathcal{L}) \subset R$ and let $\mathcal{R}^{(I)}$ denote its symbolic Rees algebra. Let $D_1, \ldots, D_n \in I^{(n-1)}$ and $E \in I^{(n(n-1)-1)}$ be as above. Let $\mathbf{X} = \{X_1, \ldots, X_n\}, \mathbf{Y} = \{Y_1, \ldots, Y_{n+1}\}, \mathbf{Z} = \{Z_1, \ldots, Z_n\}, W$ denote mutually independents sets of indeterminates. Consider the surjective homomorphism of R-algebras:*

$$\pi : k[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, W] \twoheadrightarrow R[It, D_1 t^{n-1}, \ldots, D_n t^{n-1}, E t^{n(n-1)-1}]$$

such that $X_i \mapsto X_i$, $Y_j \mapsto \Delta_j t$, $Z_r \mapsto D_r t^{n-1}$ e $W \mapsto E t^{n(n-1)-1}$. Then $\ker(\pi)$ contains the following polynomials:

(1) *The entries of $\mathbf{X} \cdot B^t$ (n such polynomials)*

(2) *The entries of $\mathbf{Z} \cdot B$ (n such polynomials)*

(3) *The entries of $\mathbf{X}^t \cdot \mathbf{Z} - \mathrm{adj}(B)$ (n^2 such polynomials)*

(4) *The polynomials of the shape $\{X_1 W^{n-1} - d_1(\mathbf{Z}), \ldots, X_n W^{n-1} - d_n(\mathbf{Z})\}$, where $d_1, \ldots, d_n$ are forms defining the inverse of $D_1, \ldots, D_n$ (n such polynomials)*

(5) *The polynomials of the shape $\{Y_1 W - \delta_1(\mathbf{Z}), \ldots, Y_{n+1} W - \delta_{n+1}(\mathbf{Z})\}$, coming from (24) below ($n+1$ such polynomials).*

Proof. The first four blocks were discussed before, namely:
(1) These are equations defining the Rees algebra $R[It]$ of I on the polynomial ring $k[\mathbf{X}, \mathbf{Y}]$. Since $R[It]$ is a subalgebra of $R[It, D_1t^{n-1}, \ldots, D_nt^{n-1}, Et^{n(n-1)-1}]$, then the equations obviously vanish under π.
(2) Note that the matrix B evaluated by $Y_j \mapsto \Delta_j t$ is a syzygy matrix of $\{D_1, \ldots, D_n\}$ by Theorem 2.18 (iv). Since Z_i maps to D_it^{n-1} the vanishing of $I_1(\mathbf{Z}, \cdot B)$ is clear as well by the same token.
(3) One argues as in the previous item based on the proof of Theorem 2.18 (iv).
(4) These equations under π just express the fact that $G = E^{n-1}$ is inversion factor of the Cremona map defined by $\{D_1, \ldots, D_n\}$.
(5) To discuss these equations, recall the relation obtained in (12):

$$\delta_j(B^t_{n1}(\mathbf{\Delta}), \ldots, B^t_{nn}(\mathbf{\Delta})) \quad = \quad E_n(\mathbf{\Delta})\Delta_j \tag{23}$$

On the other hand, we have

$$\begin{aligned} \delta_j(B^t_{n1}(\mathbf{\Delta}), \ldots, B^t_{nn}(\mathbf{\Delta})) &= \delta_j(B_{n1}(\mathbf{\Delta}), \ldots, B_{nn}(\mathbf{\Delta})) \\ &= \delta_j(X_nD_1, \ldots, X_nD_n) \\ &= X_n^n\delta_j(D_1, \ldots, D_n). \end{aligned}$$

Therefore,

$$E_n(\mathbf{\Delta})\Delta_j = X_n^n\delta_j(\mathbf{D})$$

Collecting the two resulting expressions yields

$$X_n^n\Delta_jE = X_n^n\delta_j(\mathbf{D})$$

and hence

$$\Delta_jE = \delta_j(\mathbf{D}) \tag{24}$$

as was to be shown. $\square$

We note that the intended generator of symbolic order $n(n-1)-1$ is E and not its $(n-1)$th power G; this raises a suspicion as to whether the polynomials of type (4) above are minimal generators of $\ker(\pi)$. And indeed, we have the following tightening result:

Proposition 2.21. *Keeping the notation of the previous lemma, in the generation of the ideal* $\ker(\pi)$ *one may replace the* n *equations of the form* $X_iW^{n-1} - d_i(\mathbf{Z})$ *by* n *equations of the form* $X_iW - Q_i(\mathbf{Y}, \mathbf{Z})$*, where* $Q_i(\mathbf{Y}, \mathbf{Z})$ *is a polynomial in* $k[\mathbf{Y}, \mathbf{Z}]$ *of the shape*

$$Q_i(\mathbf{Y}, \mathbf{Z}) = \sum_{t_1+\ldots+t_{n-2}=n-2} Y_{j_1}^{t_1}Y_{j_2}^{t_2}\cdots Y_{j_{n-2}}^{t_{n-2}}P_{t_1,\ldots,t_{n-2}}(\mathbf{Z}). \tag{25}$$

In particular, $(\mathbf{X})E \subset I^{n-2}\,(I^{(n-1)})^{n-1}$.

Proof. Let as above $\delta_1 = \delta_1(\mathbf{Z}), \ldots, \delta_{n+1} = \delta_{n+1}(\mathbf{Z})$ denote the n-minors of the matrix $\mathcal{L}'$ and let π as be as given.

We claim that for any collection of non-negative integers $t_1, \dots, t_s$, with $s \le n+1$, and for every subset $\{j_1, \dots, j_s\} \subset \{1, \dots, n+1\}$, the polynomials

$$Y_{j_1}^{t_1} \cdots Y_{j_s}^{t_s} W^{t_1+\dots+t_s} - \delta_{j_1}(\mathbf{Z})^{t_1} \cdots \delta_{j_s}(\mathbf{Z})^{t_s} \in \ker(\pi)$$

belong to $\ker(\pi)$.

We proceed by induction on s.

The result is clear for $s = 1$ because $Y_j W - \delta_j \in \ker(\pi)$ by the previous lemma and is a factor of $Y_j^t W^t - {\delta_j}^t$, for any t.

Thus, assume that $s > 1$ and that, without loss of generality, $t_1 \neq 0$ (the result is trivially satisfied if all t's are null). Write

$$\begin{aligned}
\left(Y_{j_1}^{t_1} W^{t_1} - \delta_{j_1}^{t_1}\right) \; Y_{j_2}^{t_2} \cdots Y_{j_s}^{t_s} W^{t_2+\dots+t_s} &= Y_{j_1}^{t_1} Y_{j_2}^{t_2} \cdots Y_{j_s}^{t_s} W^{t_1+\dots+t_s} - \delta_{j_1}^{t_1} Y_{j_2}^{t_2} \cdots Y_{j_s}^{t_s} W^{t_2+\dots+t_s} \\
&= Y_{j_1}^{t_1} Y_{j_2}^{t_2} \cdots Y_{j_s}^{t_s} W^{t_1+\dots+t_s} - \delta_{j_1}^{t_1} \cdots \delta_{j_s}^{t_s} + \delta_{j_1}^{t_1} \cdots \delta_{j_s}^{t_s} - \delta_{j_1}^{t_1} Y_{j_2}^{t_2} \cdots Y_{j_s}^{t_s} W^{t_2+\dots+t_s} \\
&= \left(Y_{j_1}^{t_1} Y_{j_2}^{t_2} \cdots Y_{j_s}^{t_s} W^{t_1+\dots+t_s} - \delta_{j_1}^{t_1} \cdots \delta_{j_s}^{t_s}\right) - \delta_{j_1}^{t_1} \left(Y_{j_2}^{t_2} \cdots Y_{j_s}^{t_s} W^{t_2+\dots+t_s} - \delta_{j_2}^{t_2} \cdots \delta_{j_s}^{t_s}\right)
\end{aligned}$$

Applying the inductive hypothesis on the two ends of this strand of inequalities shows that the polynomial

$$Y_{j_1}^{t_1} Y_{j_2}^{t_2} \cdots Y_{j_s}^{t_s} W^{t_1+\dots+t_s} - \delta_{j_1}(\mathbf{Z})^{t_1} \cdots \delta_{j_s}(\mathbf{Z})^{t_s}$$

also belongs to $\ker(\pi)$. In particular, taking $s = n-2$ and $t_1, \dots, t_{n-2}$ any partition of $n-2$, the polynomial

$$Y_{j_1}^{t_1} Y_{j_2}^{t_2} \cdots Y_{j_{n-2}}^{t_{n-2}} W^{n-2} - \delta_{j_1}(\mathbf{Z})^{t_1} \cdots \delta_{j_{n-2}}(\mathbf{Z})^{t_{n-2}} \tag{26}$$

belongs to $\ker(\pi)$.

On the other hand, as seen in Theorem 2.18 (ii), the coordinate forms $\{d_1 = d_1(\mathbf{Z}), \dots, d_n = d_n(\mathbf{Z})\}$ defining the inverse of the Cremona map defined by $\{D_1, \dots, D_n\}$ also constitute a complete set of source inversion factors of the birational map defined by the n-minors $\delta_1, \dots, \delta_n$ of the general linear matrix $\mathcal{L}'$. Therefore, Proposition 2.9 gives

$$(d_1, \dots, d_n) \subset (\delta_1, \dots, \delta_{n+1})^{(n-1)}$$

and, for even more reason

$$(d_1, \dots, d_n) \subset (\delta_1, \dots, \delta_{n+1})^{(n-2)} = (\delta_1, \dots, \delta_{n+1})^{n-2}. \tag{27}$$

Fixing $i \in \{1, \dots, n\}$ we can write

$$d_i(\mathbf{Z}) = \sum_{t_1+\dots+t_{n-2}=n-2} P_{t_1,\dots,t_{n-2}}(\mathbf{Z}) \delta_{j_1}(\mathbf{Z})^{t_1} \cdots \delta_{j_{n-2}}(\mathbf{Z})^{t_{n-2}}. \tag{28}$$

Thus, one gets that the polynomial

$$\begin{aligned}
X_i W^{n-1} - d_i(\mathbf{Z}) \quad &- \sum_{t_1+\dots+t_{n-2}=n-2} P_{t_1,\dots,t_{n-2}}(\mathbf{Z}) \left(Y_{j_1}^{t_1} Y_{j_2}^{t_2} \cdots Y_{j_{n-2}}^{t_{n-2}} W^{n-2} - \delta_{j_1}(\mathbf{Z})^{t_1} \cdots \delta_{j_{n-2}}(\mathbf{Z})^{t_{n-2}}\right) \\
&= W^{n-2} \left(X_i W - \sum_{t_1+\dots+t_{n-2}=n-2} P_{t_1,\dots,t_{n-2}}(\mathbf{Z}) Y_{j_1}^{t_1} Y_{j_2}^{t_2} \cdots Y_{j_{n-2}}^{t_{n-2}}\right)
\end{aligned}$$

belongs to $\ker(\pi)$. Since $\ker(\pi)$ is a prime ideal and $W \notin \ker(\pi)$, we conclude that

$$X_i W - \sum_{t_1+\dots+t_{n-2}=n-2} P_{t_1,\dots,t_{n-2}}(\mathbf{Z}) Y_{j_1}^{t_1} Y_{j_2}^{t_2} \cdots Y_{j_{n-2}}^{t_{n-2}} \in \ker(\pi)$$

as was to be shown.

The second statement is clear. $\square$

We now come to the main result of this part.

Theorem 2.22. (char(k) = 0) *Let $\mathcal{L}$ denote an $(n+1)\times n$ general linear matrix over $R = k[X_1,\ldots,X_n]$, with $n\geq 3$. Set $I := I_{n-1}(\mathcal{L}) \subset R$ and let $\mathcal{R}^{(I)}$ denote its symbolic Rees algebra.*

(i) *Let $\pi : R[\mathbf{Y},\mathbf{Z},W] \twoheadrightarrow R[It, D_1t^{n-1},\ldots,D_nt^{n-1},Et^{n(n-1)-1}]$ stand for the R-algebra homomorphism as defined in* Lemma 2.20. *Then*

(a) *The kernel of π is generated by the polynomials*

$I_1(\mathbf{X}\cdot B^t),\ I_1(\mathbf{Z}\cdot B),\ I_1(\mathbf{X}^t\cdot \mathbf{Z} - \mathrm{adj}(B)),\ Y_jW - \delta_j(\mathbf{Z})\ (1\leq j\leq n+1),\ X_iW - Q_i(\mathbf{Y},\mathbf{Z})\ (1\leq i\leq n),$

where $Q_i(\mathbf{Y},\mathbf{Z})$ is described in Proposition 2.21.

(b) $\mathcal{R}^{(I)} = R[It, D_1t^{n-1},\ldots,D_nt^{n-1},Et^{n(n-1)-1}]$

(ii) *$\mathcal{R}^{(I)}$ is a Gorenstein normal domain.*

Proof. (i) We first claim that W is a nonzerodivisor on $R[\mathbf{Y},\mathbf{Z},W]/\mathcal{P}$. For this, we will use Gröbner basis theory. Namely, consider the degrevlex order with $\mathbf{Z} > \mathbf{Y} > \mathbf{X} > W$. As is well-known, it suffices to show that W is not a factor of a minimal generator of in($\mathcal{P}$). Now, none of the monomials $Y_jW\ (1\leq j\leq n+1)$, $X_iW\ (1\leq i\leq n)$ is a minimal generator of in($\mathcal{P}$) since the order first breaks a tie by the degree, while both $\delta_j(\mathbf{Z})$, $Q_i(\mathbf{Y},\mathbf{Z})$ have degree at least $n\geq 3$. However, a multiple thereof could be a fresh generator of in($\mathcal{P}$). We must exclude this possibility.

On the other hand, any fresh initial generator is found by an iteration of the so-called S-polynomials ([25, Section 1.2]) associated to pairs of elements of $\mathcal{P}$ starting out with pairs of the given set of generators thereof. Since any generator coming from the part $I_1(\mathbf{X}\cdot B^t)$, $I_1(\mathbf{Z}\cdot B)$, $I_1(\mathbf{X}^t\cdot\mathbf{Z} - \mathrm{adj}(B))$ does not involve W, we must use at least one among the equations $Y_jW\ (1\leq j\leq n+1)$, $X_iW\ (1\leq i\leq n)$. We now analyze the nature of such S-polynomials and their iterations according to the given starting pair of generators of $\mathcal{P}$.

(1) Starting pair $\{Y_jW - \delta_j(\mathbf{Z}),\ Y_kW - \delta_k(\mathbf{Z})\}$ $(j\neq k)$

Consider the respective initial terms, which are pure monomials in $\mathbf{Z}$ – this is because, as already remarked, $\deg(\delta_j(\mathbf{Z})) = n > 2$. Say, $M_j = M_j(\mathbf{Z}), M_k = M_k(\mathbf{Z})$ are the respective initial terms and set $H := \gcd(M_j,M_k)$, so $M_j = N_jH$, $M_k = N_kH$, with $\gcd(N_j,N_k) = 1$. Then the associated S-polynomial has the shape

$$S := (N_kY_j - N_jY_k)W - (N_k\delta'_j - N_j\delta'_k), \tag{29}$$

where $\delta_j = M_j + \delta'_j, \delta_k = M_k + \delta'_k$. By a similar token, $\deg(N_k\delta'_j) = \deg(N_j\delta'_k) > \deg(N_kY_jW) = \deg(N_jY_kW)$, and hence the initial term of (29) has to come from either $N_k\delta'_j$ or $N_j\delta'_k$, provided we make sure that the $N_k\delta'_j - N_j\delta'_k$ does not vanish. But since $N_jM_k - N_kM_j = 0$ by construction, this vanishing would imply the relation $N_k\delta_j - N_j\delta_k = 0$. However, δ_j,δ_k are distinct n-minors of a general linear matrix, hence are relatively prime. This would force a trivial relation, hence N_j would be multiples of δ_j – this is absurd since N_j is a monomial.

Repeat the S-polynomial procedure using (29) and any other equation of type $Y_pW - \delta_p$ obtaining a new S-polynomial. To make it explicit, say, the initial term of old S comes from

$N_k\delta'_j$; then write $\delta'_j = M'_j + \delta''_j$, where M'_j is the initial term. Also write, as above, $\delta_p = M_p + \delta'_p$, with M_p its initial term. Finally, set $M'_j = N'_j H$, $M_p = N'_p H$, where $\gcd(N'_j, N'_p) = 1$. Then the updated S-polynomial is

$$\begin{aligned} S' := & \Big(N'_p(N_kY_j - N_jY_k) - N_kN'_jY_p\Big)W \\ & - \Big(N'_p(N_k\delta''_j) - N'_p(N_j\delta'_k) - N_k(N'_j\delta'_p)\Big). \end{aligned}$$

Counting degrees as before, we that the degree of the top part is lower than that of the bottom part. Therefore, the initial term of S' will come off the bottom part provided we show it does not vanish. Supposing this were the case, using the basic S-pair relation $N'_p(N_kM'_j) - (N_kN'_j)M_p$, we get the relation $N'_p(N_k\delta'_j - N_j\delta'_k) - N_kN'_j\delta_p = 0$. Since $\gcd(N'_j, N'_p) = 1$, we see that $N'p$ divides $N_k\delta_p$. But δ_p is the n-minor of a general linear matrix, hence N'_p divides N_k. Substituting in the previous relation and simplifying yields $N_k\delta'_j - N_j\delta'_k = N\delta_p$, for some monomial $N \in k[\mathbf{Z}]$. But this implies that our initial S-polynomial in (29) has the form $(N_kY_j - N_jY_k)W - N\delta_p$. Using $Y_p - \delta_p$, one gets $(N_kY_j - N_jY_k - NY_p)W = 0$ and hence $N_kY_j - N_jY_k - NY_p = 0$. This is nonsense since N_k, N_j, N are polynomials in $k[\mathbf{Z}]$.

Now the general iterated step is clear, therefore the iteration of S-polynomials using only this packet of equations gives fresh initial generators which are monomials in $\mathbf{Z}$ exclusively.

(2) Starting pair $\{X_iW - Q_i(\mathbf{Y},\mathbf{Z}),\ X_lW - Q_l(\mathbf{Y},\mathbf{Z})\}$ $(i \neq l)$

The argumentative strategy is analogous to the one in the previous case: write

$$\left\{ \begin{array}{ll} Q_i = M_i + Q'_i, & Q_l = M_l + Q'_l \\ M_i = M'_iH, & M_l = M'_lH, \end{array} \right.$$

where $M_i = \mathrm{in}(Q_i)$, $M_l = \mathrm{in}(Q_l)$ and $\gcd(M'_i, M'_l) = 1$. Note that, from (28) and since $\deg(d_i(\mathbf{Z})) = n(n-1) - 1$ and $\deg(\delta(\mathbf{Z})) = n(n-2)$, one has $\deg(Q_i) = 2n - 3$. Then the resulting polynomial is the sum of two homogeneous polynomials

$$S := (M'_lX_i - M'_iX_l)W - (M'_lQ'_i - M'_iQ'_l),$$

where $\deg(M'_lX_iW) = 2n-3-h+2 = 2n-1-h$, $\deg(M'_lQ'_i) = 2n-3-h+2n-3 = 4n-6-h$, with $h = \deg(H)$. Again, we have the strict inequality $4n-6 > 2n-1$, for $n \geq 3$. To show that the initial term of the above polynomial belongs to the rightmost polynomial we need to know that the latter does not vanish. But if it did, then we would have the equality $M'_lQ_i = M'_iQ_l$, where $\gcd(M'_i, M'_l) = 1$. Now, since the multipliers M'_i, M'_l are relatively prime then M'_i is a factor of Q_i. By (28), evaluating we would get that d_i has a monomial factor in $k[\mathbf{Z}]$; this is ruled out by fact that d_i is a coordinate function of the inverse map to the Cremona map defined by $\{D_1, \ldots D_n\}$ which are sufficiently general forms.

We can now iterate as in case (1). Thus, let

$$S_X = \Big(\sum_i N_i(\mathbf{Y},\mathbf{Z})X_i\Big)W - P(\mathbf{Y},\mathbf{Z}) \tag{30}$$

stand for an iterated S-polynomial out of the "X_iW" packet, with $M = M(\mathbf{Y},\mathbf{Z})$ denoting the corresponding initial term. By induction, we have $\deg(N_i) + 2 < \deg(P)$. Write $P = M + P' =: \mathrm{in}(P) + P'$ and $Q_r = M_r + Q'_r := \mathrm{in}(Q_r) + Q'_r$. Then the new S-polynomial has the shape

$$\sum_i M'_rN_iX_iW - M'X_rW - (M'_rP' - M'Q'_r),$$

where $M = M'H$, $M_r = M_r'H$, with $\gcd(M', M_r') = 1$. We assume that $\deg(H) > 0$ as otherwise there is nothing to prove by [25, Exercise 1.2.2]. Then $\deg(M_r'N_iX_iW) = 2n - 3 + \deg(N_i) + 2 - h < 2n - 2 + \deg(P) - h = \deg(M_r'P')$ and, similarly, $\deg(M'X_rW) = \deg(M) + 2 - h = \deg(P) + 2 - h < \deg(P) + 2n - 3 - h = \deg(M_r'P')$. Moreover, if $M_r'P' = M'Q_r'$ then $M_r'P = M'Q_r$ as well. If $\gcd(P, Q_r) = 1$ then M_r' must be a multiple of Q_r, which is impossible since $\deg(M_r') < \deg(Q_r)$ by hypothesis. Then P and Q_r must have a proper common factor. Now, since the multipliers M_r', M' are relatively prime then M_r' is a factor of Q_r. Under $Y_j \mapsto \delta_j(\mathbf{Z})$ we would get that d_r has a monomial factor in $k[\mathbf{Z}]$; this is again ruled out by fact that d_r is a coordinate function of the inverse map to the Cremona map defined by $\{D_1, \ldots D_n\}$ which are fairly general forms.

Thus, the initial term of an S-polynomial from pairs consisting of any previous S-polynomial obtained and any other equation $X_rW - Q_r(\mathbf{Y}, \mathbf{Z})$ is a monomial in $\mathbf{Y}$ and $\mathbf{Z}$ alone.

(3) (Mixed starting pair) One of the pairs

$$\{Y_jW - \delta_j(\mathbf{Z}),\, S_X\} \quad \text{or} \quad \{X_iW - Q_i(\mathbf{Y}, \mathbf{Z}),\, S_Y\},$$

for some $1 \le j \le n+1$ and some $1 \le i \le n$, where S_Y (respectively, S_X) is any S-polynomial from the "YW" packet (respectively, from the "XW" packet).

Let us deal with these pairs separately. For the first pair, let S_X have the expression as in (30). Then the new S-polynomial has the form

$$\sum_i M_j'N_iX_iW - M'Y_jW - (M_j'P' - M'\delta_j'),$$

where $\gcd(M', M_j') = 1$. Degree counting gives $\deg(M_j'N_iX_iW) = n + \deg(N_i) + 2 - h < n + \deg(P) - h$. Moreover, vanishing of the rightmost polynomials would lead to $M_j'P = M'\delta_j$. As before, we are forced to conclude that δ_j has a factor which is a monomial. But this is impossible since δ_j is an n-minor of a general linear matrix.

For the pair of the second kind, let

$$S_Y = (\sum_j N_jY_j)W - \sum_j N_j\delta_j^{<s_j>}$$

denote an S-polynomial as iterated from the "Y_jW" packet. Here $\delta_j^{<s_j>}$ denotes a suitable summand of δ_j and the initial term of S_Y. Form the S-polynomial with some $X_iW - Q_i$, $Q_i = Q_i(\mathbf{Y}, \mathbf{Z})$, getting:

$$M_i'(\sum_j N_jY_j)W - M_{j_0}'X_iW - \Big(M_i'\sum_j N_j\delta_j^{<s_j'>} - M_{j_0}'Q_i'\Big),$$

where

$$\left\{\begin{array}{rl} M_{j_0} = & \mathrm{in}(\delta_j^{<s_{j_0}>}) \\ M_i = & \mathrm{in}(Q_i) \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rl} N_{j_0}M_{j_0} = & M_{j_0}'H \\ M_i = & M_i'H \end{array}\right.$$

with $\gcd(M_{j_0}', M_i') = 1$ and $\delta_j^{<s_j'>}$ are the updated summands of δ_j. Once again, an immediate degree count tells us that the initial term of the new S-polynomial comes from the right most

difference above, as long as the latter does not vanish. Supposing it did, we would as before get the relation $M_i' \sum_j N_j \delta_j^{<s_j>} = M_{j_0}' Q_i$, with monomial multipliers relatively prime. This implies that M_i' is a factor of Q_i, which leads to a relation $\sum_j N_j \delta_j^{<s_j>} = M_{j_0}' Q_i''$ with, say, $Q_i = M_i Q_i''$. Substituting back in S_Y gives $S_Y = (\sum_j N_j Y_j)W - M_{j_0}' Q_i''$. Multiplying S_Y by M_i and $X_i W - Q_i$ by M_{j_0}', and subtracting yields $(\sum_j M_i' N_j Y_j - M_{j_0}' X_i)W = 0$. Therefore, $\sum_j M_i' N_j Y_j = M_{j_0}' X_i$, which implies that M_{j_0}' belong to the ideal generated by a nonempty subset of the of the $\mathbf{Y}$ variables; this is absurd since $M_{j_0}' \in k[\mathbf{Z}]$.

To conclude these cases, note that iterating these two types of S-polynomials, we obtain similarly that any pair $\{S_Y, S_X\}$ yields an S-polynomial whose initial term is not divisible by W.

(4) Starting pair $\{Y_j W - \delta_j(\mathbf{Z}),\, q\}$

Here q is a generator out of $I_1(\mathbf{X} \cdot B^t),\ I_1(\mathbf{Z} \cdot B),\ I_1(\mathbf{X}^t \cdot \mathbf{Z} - \mathrm{adj}(B))$.

Let q come from $I_1(\mathbf{X} \cdot B^t)$. Then its initial term is of the form $\alpha Y_k X_i$. Since the initial term of $Y_j W - \delta_j(\mathbf{Z})$ is a monomial in $\mathbf{Z}$ alone, these two monomials are relatively prime. Therefore, the resulting S-polynomial reduces to zero relative to the pair $\{Y_j W - \delta_j(\mathbf{Z}),\, q\}$ ([25, Exercise 1.2.2]) and hence, produces no fresh initial generator.

Assume now that q comes from the packet $I_1(\mathbf{Z} \cdot B)$. By a similar token, the initial term of q has the form $\beta Z_i Y_k$. Since the initial term of $Y_j W - \delta_j(\mathbf{Z})$ is a monomial in $\mathbf{Z}$ alone, the only way to get away from reducing to zero as before is that βZ_i divide this $\mathbf{Z}$-monomial. Thus, let $M(\mathbf{Z}) Z_i$ denote the initial term of $\delta_j(\mathbf{Z})$ and write $\delta_j = \beta M(\mathbf{Z}) Z_i + P(\mathbf{Z})$. Then the resulting polynomial is

$$S := Y_k Y_j W - Y_k P(\mathbf{Z}) + M(\mathbf{Z}) q(\mathbf{Z}, \mathbf{Y}),$$

where $q(\mathbf{Z}, \mathbf{Y})$ is a 2-form of bidegree $(1,1)$ in $\mathbf{Z}, \mathbf{Y}$. Clearly, $3 = \deg(Y_k Y_j W) < 1 + n = \deg(Y_k P(\mathbf{Z})) = \deg(M(\mathbf{Z}) q(\mathbf{Z}, \mathbf{Y}))$, hence the initial term of S involves only $\mathbf{Z}$ and $\mathbf{Y}$ variables provided we show that $-Y_k P(\mathbf{Z}) + M(\mathbf{Z}) q(\mathbf{Z}, \mathbf{Y})$ does not vanish. Now, a similar reasoning as employed at the end of the argument of (1), shows that this vanishing entails a monomial syzygy between $\delta_j(\mathbf{Z})$ and the quadric q with relatively prime multipliers. This then forces $\delta_j(\mathbf{Z})$ and q to have a common factor. But q is bihomogenous, so a common factor would have to be a variable Z_l. On the other hand, $\delta_j(\mathbf{Z})$ is a minor of a general linear matrix, so cannot admit such a factor.

Keeping the essential shape of the S-polynomial obtained, namely, $S = Y_k Y_j - P(\mathbf{Y}, \mathbf{Z})$, with $P(\mathbf{Y}, \mathbf{Z})$ homogeneous of degree $n+1$ involving effectively both $\mathbf{Y}$ and $\mathbf{Z}$ variables, let us iterate with the pair $\{S, Y_r W - \delta_r(\mathbf{Z})\}$, for given $1 \leq r \leq n+1$. Write $P(\mathbf{Y}, \mathbf{Z}) = M(\mathbf{Y}, \mathbf{Z}) + P'(\mathbf{Y}, \mathbf{Z})$, $\delta_r(\mathbf{Z}) = N(\mathbf{Z}) + \delta_r'(\mathbf{Z})$, where $M = M(\mathbf{Y}, \mathbf{Z}), N = N(\mathbf{Z})$ are the respective initial terms, and $M = M'H, N = N'H$, with $\gcd(M', N') = 1$. Then the new S-polynomial is

$$\big(N' Y_k Y_j - M' Y_r\big)W - (N' P' - M' \delta_r'),$$

where the leftmost polynomial is homogeneous of degree $n + 3 - h$, with $h = \deg(H)$, while the rightmost polynomial has degree $2n + 1 - h > n + 3 - h$, for $n \geq 3$. On the other hand, the rightmost polynomial is nonzero because, otherwise, it would imply that $N'P = M'\delta_r$. Since N', M' are relatively prime, δ_r would be a multiple of $P = P(\mathbf{Y}, \mathbf{Z})$. But this is absurd since $\delta_r \in k[\mathbf{Z}]$ while $P \notin k[\mathbf{Z}]$. Therefore, the initial term of the updated S-polynomial comes from the rightmost polynomial and does not involve W. The inductive procedure is now clear: the "new" S-polynomial is a sum of two polynomials, the first involving W and degree growing like

$(s-1)n+3-t$, for $s \geq 2$ and some $t \geq 0$, the second a nonzero polynomial involving effectively the variables $\mathbf{Y}, \mathbf{Z}$ and with degree growing like $sn+1-t > (s-1)n+3-t$ (for $n \geq 3$).

Finally, consider the case where q comes from the packet $I_1(\mathbf{X}^t \cdot \mathbf{Z} - \mathrm{adj}(B))$. If $n \geq 4$, the initial term is decided by degree and has to come from some cofactor of B – the latter having degree $n-1 \geq 3 > 2 = \deg(X_i Z_l)$, for any choice of i, l. In this case, once again, the S-polynomial reduces to zero. Finally, let $n = 3$. Since we are assuming the revlex order upon monomials of same degree, the initial term of P comes from a cofactor of B, so we are done again.

Remark 2.23. To close this case, we ought to consider the S-polynomial from the pair consisting of a polynomial of the "$Y_j W$" packet and some previous S-polynomial among one of the three kinds. But, as we have seen, the only iterated S-polynomials that play any role come from the pairs

$$\{Y_j W - \delta_j(\mathbf{Z}),\, q \in I_1(\mathbf{Z} \cdot B)\}.$$

One can see that this iteration follows a pattern analogous to the first iterate, in which the initial term lives in $k[\mathbf{Y}, \mathbf{Z}]$.

(5) Starting pair $\{X_i W - Q_i(\mathbf{Y}, \mathbf{Z}),\, q\}$
Here q is again a generator out of $I_1(\mathbf{X} \cdot B^t),\ I_1(\mathbf{Z} \cdot B),\ I_1(\mathbf{X}^t \cdot \mathbf{Z} - \mathrm{adj}(B))$.

The initial term of $X_i W - Q_i(\mathbf{Y}, \mathbf{Z})$ involves both $\mathbf{Y}$ and $\mathbf{Z}$. This breaks the symmetry with respect to the discussion in case (4).

Let first q come from $I_1(\mathbf{X} \cdot B^t)$. Then $q = \alpha Y_j X_l + q'$, where $\mathrm{in}(q) = \alpha Y_j X_l$. Note that, due to the nature of B^t as matrix of partial $\mathbf{X}$-derivatives of minors of a general linear matrix, q' is a nonzero from of bidegree $(1,1)$. Write as before $Q_i = M_i + Q_i'$, where $\mathrm{in}(Q_i) = M_i = N_i \cdot \alpha Y_j$. Clearly, X_l does not divide N_i. The resulting S-polynomial is $X_l X_i W - (X_l Q_i' - N_i q')$. One has $\deg(X_l Q_i' - N_i q') = 1 + 2n - 3 = 2n - 3 - 1 + 2 = 2n - 2 > 3$, for $n \geq 3$. Moreover, if $X_l Q_i' = N_i q'$ then $X_l Q_i = N_i q$. This forces X_l to be a factor of q, which implies that q is monomial, contradicting its nature as pointed out.

Now assume that q come from $I_1(\mathbf{Z} \cdot B)$. Then $q = \beta Z_k Y_j + q'$, where $\mathrm{in}(q) = \beta Z_k Y_j$. The same remarks about the nature of q hold as above. Keeping the same notation, $Q_i = M_i + Q_i'$, where $\mathrm{in}(Q_i) = M_i$. If $Z_k Y_j$ divides M_i altogether, then the resulting polynomial is of the form $X_i W - (Q_i' - P_i q')$, for suitable $P_i \in k[\mathbf{Y}, \mathbf{Z}]$ homogeneous of degree $2n-3-2+2 = 2n-3 > 2$. Thus, we may assume that either Z_k divides M_i and Y_j does not divide M_i, or vice versa. Although the role of $\mathbf{Y}$ and $\mathbf{Z}$ are not quite symmetric in the data, the pattern is pretty much the same (and much the same as the previous case). Say, $M_i = N_i \cdot \beta Z_k$, with Y_j not dividing M_i. The resulting S-polynomial is $Y_j X_i - (Y_j Q_i' - N_i q')$. Again the inequality $2n - 2 > 3$ says that the initial term is part of $Y_j Q_i' - N_i q'$. Moreover, $Y_j Q_i = N_i q$ would imply that Y_j divide q, again a contradiction.

Finally, we settle the last case where q comes from the packet $I_1(\mathbf{X}^t \cdot \mathbf{Z} - \mathrm{adj}(B))$. If $n \geq 4$, the initial term is decided by degree and has to come from some cofactor of B – the latter having degree $n - 1 \geq 3 > 2 = \deg(X_i Z_l)$, for any i, l. Say, $q = C(\mathbf{Y}) + q'$, with $\mathrm{in}(q) = C = C(\mathbf{Y})$ of degree $n-1$. As before, $Q_i = M_i + Q_i'$, where $\mathrm{in}(Q_i) = M_i$. Set $C = C'H, M_i = N_i H, \gcd(C', N_i) = 1$. The resulting S-polynomial is $C' X_i W - (C' Q_i' - N_i q')$, where $\deg(C' Q_i' - N_i q') = n - 1 + 2n - 3 = 3n - 4 > n + 1 = \deg(C' X_i W)$. Furthermore, if $C' Q_i = N_i q$ would imply that C' divide q; this is absurd since q is of the form $X_i Z_l - p(\mathbf{Y})$.

At last, let $n=3$. Since we are assuming the revlex order upon monomials of same degree, the initial term of P comes from a cofactor of B, so we are done again.

To close this item, we refer to Remark 2.23, with the difference that here one has to consider iterated S-polynomials form all three kinds, as none reduces to zero right at the outset.

(a) Let $\mathcal{P} \subset R[\mathbf{Y}, \mathbf{Z}, W]$ denote the ideal generated by those many equations in the statement. By Lemma 2.20 and Proposition 2.21, we have $\mathcal{P} \subset \ker(\pi)$. The two ideals have same codimension: $2n+1$. Indeed, the algebra $R[It, D_1t^{n-1}, \ldots, D_nt^{n-1}, Et^{n(n-1)-1}]$ has the same dimension as the Rees algebra $R[It]$, which is $n+1$; this shows that $\ker(\pi)$ has codimension $2n+1$. As for $\mathcal{P}$, we localize at the powers of W. Then $\mathcal{P}$ and $\mathcal{P}[W^{-1}] \subset k[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, W, W^{-1}]$ have the same codimension. But in the latter the generators

$$\{Y_j - W^{-1}\delta_j(\mathbf{Z}),\, X_i - W^{-1}Q_i(\mathbf{Y}, \mathbf{Z}) \,|\, 1 \leq j \leq n+1, 1 \leq i \leq n\}$$

form a regular sequence of length $n+1+n = 2n+1$.

Therefore, to show that $\mathcal{P} = \ker(\pi)$ it suffices to prove that $\mathcal{P}$ is a prime ideal. By localizing at the powers of W, one gets an isomorphism of k-algebras

$$k[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, W, W^{-1}]/\mathcal{P}[W^{-1}] \simeq k[\mathbf{Z}, W, W^{-1}]/\widetilde{\mathcal{P}[W^{-1}]} \tag{31}$$

by mapping $X_i \mapsto W^{-1}Q_i(\mathbf{Y}, \mathbf{Z})$ and subsequently $Y_j \mapsto W^{-1}\delta_j(\mathbf{Z})$. Since $k[\mathbf{Z}, W, W^{-1}]$ has dimension $n+1$, we must conclude that $\widetilde{\mathcal{P}[W^{-1}]} = 0$. Therefore, $\mathcal{P}[W^{-1}]$ is a prime ideal, and hence so is $\mathcal{P}$.

(b) We first claim that $R[It, D_1t^{n-1}, \ldots, D_nt^{n-1}, Et^{n(n-1)-1}]$ is Cohen–Macaulay. By (i), we know that $\ker(\pi) = \mathcal{P}$. Then we argue as in (i), namely, localizing at the powers of the nonzero element $Et^{n(n-1)-1}$ gives an isomorphism onto the k-algebra $k[\mathbf{Z}, W, W^{-1}]$, which is a Cohen–Macaulay ring. Therefore, $A := R[It, D_1t^{n-1}, \ldots, D_nt^{n-1}, Et^{n(n-1)-1}]$ is Cohen–Macaulay as well.

To complete the proof of the item, we apply the criterion of Vasconcelos ([24, Propositions 7.1.4 and 10.5.1]) mentioned in the first section of this paper. For this we note that, because of Proposition 2.4 (ii) (hence the need for characteristic zero at this point), the entire symbolic algebra of I is the ideal transform of the Rees algebra of I with respect to the ideal $(\mathbf{X})$. Moreover, since A is Cohen–Macaulay, it suffices to prove that the codimension of the extended ideal $(\mathbf{X})A$ is at least 2. But this codimension is the difference of the codimension of $(\mathbf{X}, \ker(\pi))$ and that of $\ker(\pi)$. Thus, it suffices to prove that the first of these is at least $2n+3$. For it note that the image of $(\mathbf{X})$ by the isomorphism (31) is the ideal

$$\big(Q_1(W^{-1}\delta_j(\mathbf{Z}), \mathbf{Z}),\, \ldots,\, Q_n(W^{-1}\delta_j(\mathbf{Z}), \mathbf{Z})\,\big).$$

Therefore, it suffices to show that this ideal has codimension at least 2. By homogeneity we can pull out W^{-1}. Then (28) shows that this ideal is generated by the coordinate functions $\{d_1, \ldots, d_n\}$ of the inverse map of the Cremona map defined by $D_1, \ldots, D_n$. By construction, these forms have trivial gcd.

(ii) By the previous item, the symbolic algebra is Cohen–Macaulay and finitely generated. Therefore, the conclusion follows from [20]. $\square$

Remark 2.24. (1) In the case $m \geq n+2$ it may happen that elements of $I^{(n-1)}$ have standard degree less than $(m-1)(n-1)-1$. The simplest such situation occurs with $n=3$ and $m=7$,

in which case $I^{(2)}$ admits 3 minimal generators of degree 10. This implies that, in this range, the inclusion $(\mathbf{X})I^{(r)} \subset I^r$ for every $r \geq 0$ fails. This is an indication that, for general values of m, n, it may difficult to guess bounds for the value of the saturation exponent, so as to have Proposition 2.4 (ii) become more precise.

(2) Computational evidence showed that in the smallest possible numerology ($n = 3, m = 5$) the behavior of the symbolic powers is quite erratic: in the range $2 \leq r \leq 5$ there are genuine generators in $I^{(r)}$. The subsequent symbolic powers have an unpredictable behavior with genuine generators creeping up on irregular intervals; we found new symbolic generators even in $I^{(23)}$. It seems reasonable to wonder whether for $m > n + 1 \geq 4$ the symbolic Rees algebra $\mathcal{R}^{(I)}$ of I is finitely generated.

3 Structured models

3.1 Generic catalecticant matrices with leap

The basic structure in this part is an $m \times (m-1)$ *r-leap catalectic* in $R = k[X_1, \ldots, X_n]$, where $1 \leq r \leq m-1$ and $n = (m-1)(r+1)$:

$$\mathcal{C} = \begin{pmatrix} X_1 & X_2 & X_3 & \ldots & X_{m-1} \\ X_{r+1} & X_{r+2} & X_{r+3} & \ldots & X_{m+r-1} \\ X_{2r+1} & X_{2r+2} & X_{2r+3} & \ldots & X_{m+2r-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{(m-1)r+1} & X_{(m-1)r+2} & X_{(m-1)r+3} & \ldots & X_{(m-1)r+(m-1)} \end{pmatrix}$$

Note that for this size of a catalectic matrix, the number n of variables is a properly factorable number, regardless of r. In this line, the number of leap catalectic matrices for a given n is the number of proper 2-factorizations of n. The extreme values $r = 1$ and $r = m-1$ yield, respectively, the ordinary Hankel matrix and the generic matrix.

One has:

Proposition 3.1. *Let $I \subset R = k[X_1, \ldots, X_n]$ stand for the ideal of $(m-1)$-minors of an $m \times (m-1)$ r-leap catalectic matrix $\mathcal{C}$ as above, with $1 \leq r \leq m-1$. Then*

(a) $\mathrm{ht}\,(I_t(\mathcal{C})) \geq m - t + 2$ *for* $1 \leq t \leq m-2$ *and* $\mathrm{ht}\,(I) = 2$.

(b) *R/I is a Cohen–Macaulay normal domain.*

(c) *I is an ideal of linear type.*

(d) *I is normally torsionfree.*

Proof. (a) The result is clear for $t = 1$, hence assume that $2 \leq t \leq m-2$. For t in this interval, consider the submatrix C_t of $\mathcal{C}$ formed by its first t columns. Then the ideal $I_t(\mathcal{C}_t)$ of maximal minors of $\mathcal{C}_t$ is prime and satisfies $\mathrm{ht}\,(I_t(\mathcal{C}_t) \geq m-1-t+2 = m-t+1$ (see, e.g., [10, Theorem 2.1 (1)]). Since the inclusion $I_t(\mathcal{C}_t) \subset I_t(\mathcal{C})$ is clearly proper for $t \geq m-2$, we are through. For $t = m-1$, $I_t(\mathcal{C}) = I$ clearly contains a regular sequence of two minors, hence its height is 2.

(b) Cohen–Macaulyness is obvious. If $Q \subset R/I$ is a prime such that $(R/I)_Q$ is not regular, then $Q \supset I_{m-2}(\mathcal{C})/I$ (see [10, Corollary 3.3 (1)]). Thus $\mathrm{ht}\,(I_{m-2}(\mathcal{C})/I) \geq m-(m-2)+2-2 \geq 2$. Therefore, R/I satisfies the Serre condition (R_1), hence R/I is a normal domain.

(c) By (a), I satisfies the condition (F_1) (or G_∞). Therefore, it is of linear type (see [14]).

(d) The assertion could possibly be derived from the methods of [17], but one can give a direct argument in the present situation. The proof is exactly as the one of Proposition 2.3 (iv), with a minor adaptation to the present environment. Namely, in that former proof the only passage where one used that the entries of the matrix were general linear forms was to quote Theorem 2.1, by which $\mathrm{ht}\, I_{t_0+1}(\mathcal{L}) = (m-t_0)(m-t_0-1)$, and hence one could deduce that $\mathrm{ht}\, I_{t_0+1}(\mathcal{L}) - 1 \geq m - t_0$. Replacing $\mathcal{L}$ by the present $\mathcal{C}$, we have $\mathrm{ht}\, I_{t_0+1}(\mathcal{C}) - 1 \geq m - t_0$ directly from item (a). The rest of the argument remains unchanged. $\square$

3.2 Generic sub-Hankel matrices

From the previous part, with $r = 1$, we know that the ideal $I = I_{m-1}(\mathcal{H})$ of an $m \times (m-1)$ Hankel matrix $\mathcal{H}$ in $n = 2(m-1)$ variables satisfies several nice properties. We now want to consider a degeneration of $\mathcal{H}$, in which a lower corner of suitable size has its entries replaced by zero. The $m \times m$ version of this model has been introduced in [5] in connection to homaloidal theory.

We set, namely:

$$\mathcal{SH} = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & \ldots & X_{n-3} & X_{n-2} \\ X_2 & X_3 & X_4 & X_5 & \ldots & X_{n-2} & X_{n-1} \\ X_3 & X_4 & X_5 & X_6 & \ldots & X_{n-1} & X_n \\ X_4 & X_5 & X_5 & X_7 & \ldots & X_n & 0 \\ . & . & . & . & \ldots & . & . \\ . & . & . & . & \ldots & . & . \\ . & . & . & . & \ldots & . & . \\ X_{n-3} & X_{n-2} & X_{n-1} & X_n & \ldots & 0 & 0 \\ X_{n-2} & X_{n-1} & X_n & 0 & \ldots & 0 & 0 \\ X_{n-1} & X_n & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{32}$$

This model satisfies all the previous properties, except normality:

Proposition 3.2. *Let $n \geq 4$ and $I := I_{n-2}(\mathcal{SH})$. Then:*

(a) $\mathrm{ht}\,(I_t(\mathcal{SH})) \geq n - t + 1$ *for* $1 \leq t \leq n-3$ *and I is a prime ideal of height* 2.

(b) *R/I is* <u>not</u> *normal.*

(c) *I is an ideal of linear type.*

(d) *I is normally torsionfree.*

Proof. (a) The proof is similar to the argument used in the catalecticant case. As before, the case $t = 1$ is immediate. Next we let $\mathcal{SH})_t$ denote the submatrix of $\mathcal{SH})$ with t columns, for values of t in the range $2 \leq t \leq n-3$. Since these matrices (including the lowest value $t = n-2$) specialize from the corresponding Hankel matrix, one gets

$$k[X_1, \ldots, X_n]/I_t(\mathcal{SH}_t) \simeq k[X_1, \ldots, X_{t+n-2}]/(X_{n+1}, \ldots, X_{t+n-2}, I_t(\mathcal{H}_t).$$

From this and [9, Theorem 1] it follows that $I_t(\mathcal{SH}_t)$ is a prime ideal of height $n-t$. In particular, with $t = n-2$ it yields the second assertion of (a). Now, for $2 \leq t \leq n-3$, the ideal $I_t(\mathcal{SH})$

is not prime (as it is not even reduced, containing a nontrivial power of X_n). Therefore, the inclusion $I_t(\mathcal{SH}_t) \subset I_t(\mathcal{SH})$ is proper since the smaller ideal is prime. This yields the main assertion of this item.

(b) We show that I does not satisfy (R_1). For this, consider the height 3 prime $P = (X_{n-2}, X_{n-1}, X_n)$. Clearly, $I \subset P$ by direct inspection on the shape of the matrix. Note that the upper left $(n-3)$-minor of $\mathcal{SH}$ has the form $X_{n-3}^{n-3} + q$ where $q \in P$, hence does not belong to I. After appropriate row/column operations, we se that $I_P = (\Delta_{n-2}, \Delta_{n-1})$, where Δ_i denotes $(n-2)$-minor of $\mathcal{SH}$ obtained by omitting the ith row. We claim that R_P/I_P is not regular. For this, it suffices to show that $\Delta_{n-2} \in P^2$. But

$$\Delta_{n-2} = (-1)^{n-1} X_{n-1} \det \begin{pmatrix} X_2 & X_3 & \dots & X_{n-2} \\ X_3 & X_4 & \dots & X_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ X_{n-2} & X_{n-1} & \dots & 0 \end{pmatrix} + (-1)^n X_n \det \begin{pmatrix} X_1 & X_3 & \dots & X_{n-2} \\ X_2 & X_4 & \dots & X_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ X_{n-3} & X_{n-1} & \dots & 0 \end{pmatrix}$$

Note that the two determinants have same last column and the nonzero entries on the column are X_{n-2}, X_{n-1}, X_n. Therefore, expanding these determinants along their last column clearly shows the claim.

(c) and (d) are proved exactly the same way as in Proposition 3.1 $\square$

3.3 Quasi-Hankel matrices

We now propose a model for a linear matrix in the case where $m \leq n$, based on the Hankel process. Note that the ordinary generic $m \times (m-1)$ Hankel matrix requires $2m-2$ variables. However, if $m \leq n \leq 2m-2$, one can still introduce a *quasi-Hankel* matrix, denoted $\mathcal{QH}_{m,n}$.

Let us deal with the border case $m = n$. Here one takes a matrix of the form

$$\mathcal{QH}_{n,n} = \begin{pmatrix} X_1 & X_2 & X_3 & \dots & X_{n-2} & X_{n-1} \\ X_2 & X_3 & X_4 & \dots & X_{n-1} & X_n \\ X_3 & X_4 & X_5 & \dots & X_n & \ell_{3,1} \\ X_4 & X_5 & X_5 & \dots & \ell_{4,1} & \ell_{4,2} \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ X_{n-1} & X_n & \ell_{n-1,1} & \dots & \ell_{n-1,n-4} & \ell_{n-1,n-3} \\ X_n & \ell_{n,1} & \ell_{n,2} & \dots & \ell_{n,n-3} & \ell_{n,n-2} \end{pmatrix}, \tag{33}$$

which is Hankel upper-triangular and the ℓ's are $\binom{n-1}{2}$ general linear forms. One can go even more unique, by setting

$$\begin{aligned} \ell_{3,1} &= X_1 - X_2 \\ \ell_{4,1} &= X_1 - X_3,\ \ell_{4,2} = X_2 - X_3 \\ \ell_{5,1} &= X_1 - X_4,\ \ell_{5,2} = X_2 - X_3,\ \ell_{5,3} = X_3 - X_4 \\ &\vdots \\ \ell_{n-1,1} &= X_1 - X_{n-2},\ \dots,\ \ell_{n-1,n-3} = X_{n-3} - X_{n-2} \\ \ell_{n,1} &= X_1 - X_{n-1},\ \dots,\ \ell_{n,n-2} = X_{n-2} - X_{n-1}. \end{aligned}$$

These choices are not strictly unique. The main guiding principles are as follows: first, on the ith row the additional linear entries should only use variables indexed by the set $\{1,...,i-1\}$; second, repeat as little as possible linear forms already used on the previous row.

If $n>m$ then the situation is even more favorable as one needs to fill in less additional linear forms. Since this model is an intermediate specialization between the Hankel $\mathcal{H}$ and the sub-Hankel $\mathcal{SH}$, it is expected that (the corresponding proofs of) the results of Proposition 3.1 extend to this case, except possibly for normality of R/I, which may require an explicit argument. Indeed, this expectation comes true as we now show, adding an extra bonus concerning the birational behavior.

Proposition 3.3. *Let $I := I_{n-1}(\mathcal{Q})$. Then:*

(a) $\mathrm{ht}\,(I_t(\mathcal{QH})) \geq n-t+2$ *in the range* $1 \leq t \leq n-2$, *while* $\mathrm{ht}\,(I) = 2$

(b) *R/I is a normal domain.*

(c) *I is an ideal of linear type.*

(d) *The rational map $\mathfrak{G} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ defined by the $(n-1)$-minors of $\mathcal{QH}$ is a Cremona map.*

(e) *For every $r \geq 0$ such that $I^{(r)} \neq I^r$, the R-module $I^{(r)}/I^r$ is $(\mathbf{X})$-primary.*

(f) *$\mathcal{R}^{(I)}$ is a Gorenstein normal domain such that $\mathcal{R}^{(I)} = R[It, Dt^{n-1}]$, where D is the source inversion factor of the Cremona map defined by the $(n-1)$-minors of $\mathcal{QH}$; if, moreover,* $\mathrm{char}(k) = 0$ *then $D = \det(\Theta)$, where Θ stands for the Jacobian matrix of the $(n-1)$-minors of $\mathcal{QH}$.*

Proof. As a preliminary, we note that $\mathcal{QH}$ is a specialization of the matrix $\mathcal{G}$ of the same order:

$$\mathcal{G} = \begin{pmatrix} X_1 & X_2 & X_3 & \ldots & X_{n-2} & X_{n-1} \\ X_2 & X_3 & X_4 & \ldots & X_{n-1} & X_n \\ X_3 & X_4 & X_5 & \ldots & X_n & Z_{3,1} \\ X_4 & X_5 & X_5 & \ldots & Z_{4,1} & Z_{4,2} \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ X_{n-1} & X_n & Z_{n-1,1} & \ldots & Z_{n-1,n-4} & Z_{n-1,n-3} \\ X_n & Z_{n,1} & Z_{n,2} & \ldots & Z_{n,n-3} & Z_{n,n-2} \end{pmatrix}, \tag{34}$$

where the $\mathbf{Z}$-entries are ordered indeterminates over R. Since $\mathcal{G}$ also specializes to the full generic Hankel matrix of the same size and the latter is 1-generic, it follows that it too is 1-generic. For every t in the range $2 \leq t \leq n-1$, let $\mathcal{G}_t$ denote the submatrix of $\mathcal{G}$ formed with the first t columns. Then, by [10, Theorem 2.1, Corollary 3.3], one has the following properties:

(i) $I_t(\mathcal{G}_t) = n-t+1$, for $2 \leq t \leq n-1$.

(ii) $I_t(\mathcal{G}_t)$ is a prime ideal.

(iii) The ideal $I_{t-1}(\mathcal{G}_t)/I_t(\mathcal{G}_t)$ defines the singular locus of $R[\mathbf{Z}]/I_t(\mathcal{G}_t)$

It follows that, for every value in the range $2 \leq t \leq n-2$, $R[\mathbf{Z}]/J_t$ is a Cohen–Macaulay ring of codimension $n-t+2$, where J_t is the ideal generated by $I_t(\mathcal{G}_t)$ and the lower right corner t-minor of $\mathcal{G}$.

Therefore, (a) follows.

(b) Using the result of (iii) above, the same argument as the proof of Proposition 2.3 (ii) takes place here.

(c) Quite generally, this follows from (a).

(d) Since I is of linear type and is linearly presented, then it defines a Cremona map by [8].

(e) Using (a), this is proved exactly as Theorem 2.4.

(f) The proof is the same as the one of Theorem 2.13 – note the part referring to the Jacobian determinant in characteristic zero follows from Proposition 2.11, which holds as soon as the ideal of minors is linearly presented and of linear type. $\square$

We close with a couple of more general questions.

Question 3.4. (1) What are natural examples of (characteristic free) perfect, codimension 2, homogeneous, prime ideals $I \subset R = k[X_1, \ldots, X_n]$, generated in fixed degree (perhaps not necessarily linearly presented), such that R/I is normal and I admits non-ordinary symbolic powers $I^{(m)}$ of order $m \leq n-2$? If "prime" is omitted (hence, so is "normal") and "perfect" is also omitted then a radical such ideal is the monomial ideal generated by the 2-paths of a pentagon and its analogues in higher dimensions; if "prime" is omitted (hence, so is "normal"), but "perfect" is required then the Jonquières–Magnus ideal ($(n-1)$-paths in an n-agon) is an example; it would be interesting to describe classes of such examples with I prime and R/I normal.

(2) Note that in the setup of this paper, the two alternatives in [25, Proposition 3.5.13] coincide set-theoretically, namely, the radical of the Jacobian ideal is read off the free presentation of the ideal. This phenomenon played a central role in the preliminaries of this work. It seems appropriate to ask when this is the case beyond the present assumptions.

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Authors' addresses:

Zaqueu Ramos, Departamento de Matemática, CCET, Universidade Federal de Sergipe
49100-000 São Cristovão, Sergipe, Brazil
e-mail: zaqueu@dmat.ufpe.br

Aron Simis, Departamento de Matemática, CCEN, Universidade Federal de Pernambuco
50740-560 Recife, PE, Brazil.
e-mail: aron@dmat.ufpe.br